A MatLab collection of variational inequality problems[*]

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Abstract

In order to give a uniform basis for testing several algorithms, in this work, we have collected a set of variational inequality problems. At the URL (http://dm.unife.it/pn2o/software.html) the MatLab M-script and M-function files related to the considered test problems are downloadable (TESTVIPs).

Introduction

We consider the classical variational inequality problem VIP(F,C), which is to find a point x^* such that

$$x^* \in C \qquad < F(x^*), x - x^* > \geq 0 \quad \forall x \in C \tag{1}$$

where C is a nonempty closed convex subset of $\Re^n$, $< .,. >$ the usual inner product in $\Re^n$ and $F : \Re^n \to \Re^n$ is a continuous function. Let C^* be the set of the solutions.
In the special case where $C = \Re^n_+$, problem (1) is a *nonlinear complementary problem* (NCP):

$$x^* \geq 0, \qquad F(x^*) \geq 0 \quad and \quad < x^*, F(x^*) >= 0. \tag{2}$$

If F is affine, $F(x) = Mx + q$ where $M \in \Re^{nxn}$ (is a positive semidefinite matrix) and $q \in \Re^n$, then the problem (1) is an affine variational inequality

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problem and (2) is a *linear complementary problem* (LCP).
In this work we collect a set of test problems arising from the literature.
For any problem, we report at least a solution and we propose an initial point to start an iterative method.
Each test problem is related to two files: the first is an M-script file where the constraints and other parameters of the problem are defined. The variable fs is put equal to the string containing the name of the M-function file containing the definition of F.
For example, the first test problem (Mathiesen's problem) is related to the following two files *inputmathiesen.m* and *mathiesen.m*:

- ```
  %script (inputmathiesen.m)
  %
  %constraint (VIP)
  n=3;                  %dimension
  A=[1,-1,-1];          %constraint (A*x<=b)
  b=0;
  Aeq=[1 1 1];          %equality constraint (Aeq*x=beq)
  beq=1;
  lb=[eps,eps,0]';      %lower bound
  ub=[];                %upper bound
  %
  %starting point
  x=[.1 .8 .1]';
  %x=[0.4, 0.3,0.3]';
  %
  %function F
  fs='mathiesen';
  ```

- ```
  %M-function (mathiesen.m)
  function [f]=mathiesen(x)
  %
  %Reference:
  ```

```
%L. Mathiesen,
% 'An algorithm based on a sequence of linear
%  complementary problems applied to a Walrasian
%  equilibrium model: an example',
%Mathematical Programming, 37 (1987), pp.1-18.
%
f=-[0.9*(5*x(2)+3*x(3))/x(1)

    0.1*(5*x(2)+3*x(3))/x(2)-5

    -3];
```

In order to use test problem it is sufficient to insert the following instruction in the code implementing a method:

```
eval('inputmathiesen')
```

1 Test Problems

1.1 Mathiesen's Problem

This problem was used first by Mathiesen [7]. The function $F : \Re^3 \rightarrow \Re^3$ is

$$\mathbf{F}(\mathbf{x}) = -\begin{bmatrix} 0.9(5x_2 + 3x_3)/x_1 \\ 0.1(5x_2 + 3x_3)/x_2 - 5 \\ -3 \end{bmatrix},$$

and its feasible set is:

$$C = \{x \in \Re^3_+ | x_1 + x_2 + x_3 = 1, \quad x_1 - x_2 - x_3 \leq 0\}.$$

This example is an Walrasian model in which the consumer demand function is determined by a single consumer; there is one production activity, and three goods.
The M-files corresponding to the problem are: *inputmathiesen.m* and *matiesen.m.*
We propose to use as starting points are $x^0 = (0.1, 0.8, 0.1)$ or $x^0 = (0.4, 0.3, 0.3)$.
A solution of the problem is $x^* = (0.5, 0.08, 0.41)$.

1.2 Kojima-Shindo's Problem

In this problem test [4], the function $F : \Re^4 \rightarrow \Re^4$ is defined as follows:

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}.$$

If the feasible set is the following simplex:

$$C = \{x \in \Re^4_+ | x_1 + x_2 + x_3 + x_4 = 4\},$$

we have a VIP.

A possible choice of the starting point is $x^0 = (2, 0, 0, 2)$ and, for this example, we can obtain as solution to this problem the point:$x^1 = (1.22, 0, 0, 0.5)$. In this case the two M-files are *inputkojshi.m* and *kojshi.m.*

If the feasible region is the nonnegavite orthant of $\Re^4$, $x \in \Re^4_+$, we have a NCP. In this case a possible choice of the starting point is $x^0 = (2, 0, 0, 2)$ and for this example we can obtain as solution the point: $x^1 = (1.22, 0, 0, 0.50)$. In this case the two M-files are *inputkojshibox.m* and *kojshi.m.*

1.3 Braess Network Problem

In [6], Marcotte considers the Braess paradox network with the separable linear cost function illustrated in Figure 1, in which the arcs are ordered as follows:

$$(1, 2), (1, 3), (2, 3), (2, 4), (3, 4).$$

The delay function is:

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{23} \\ x_{24} \\ x_{34} \end{bmatrix} + \begin{bmatrix} 0 \\ 50 \\ 10 \\ 50 \\ 0 \end{bmatrix}.$$

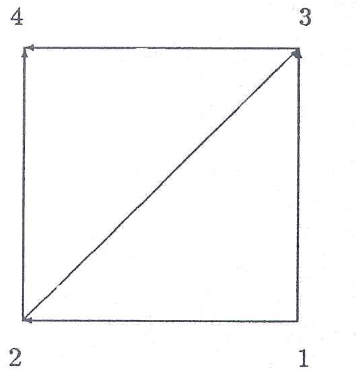

Figure 1: The Braess paradox network

In this example the feasible set is:

$$C = \{x \in \Re^5_+ | Bx = b\},$$

where the node-arc incidence matrix and b are respectively as follows:

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ -6 \end{bmatrix}.$$

This matrix has rank three; then we have considered the full row-rank $\widehat{B}$ submatrix of B, and the vector b, as follows

$$\widehat{\mathbf{B}} = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \widehat{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}.$$

A reasonable choice of the starting point is $x^0 = (6, 0, 6, 0, 6)$ and the solution is $x^* = (4, 2, 2, 2, 4)$.
In this case the M-files corresponding to the problem are *inputbraessnet.m* and *braessnet.m*.

1.4 User-Optimized Traffic Pattern

In [2], Dafermos computes user-optimization traffic pattern for the sim-

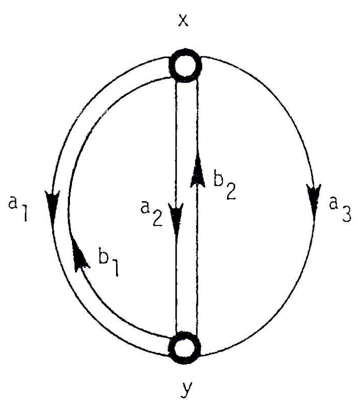

Figure 2: Network

ple network shown in Figure 2 , with only two nodes x, y and five links a_1, a_2, a_3, b_1, b_2, where a_1, a_2, a_3 are directed from x to y and b_1, b_2 are the return of a_1, a_2 respectively.
The travel cost functions are given by

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 10 & 0 & 0 & 5 & 0 \\ 0 & 15 & 0 & 0 & 5 \\ 0 & 0 & 20 & 0 & 0 \\ 2 & 0 & 0 & 20 & 0 \\ 0 & 1 & 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \\ x_{b1} \\ x_{b2} \end{bmatrix} + \begin{bmatrix} 1000 \\ 950 \\ 3000 \\ 1000 \\ 1300 \end{bmatrix};$$

further the problem is subjected to the following constraints:
$C = \{x \in \Re^5_+ | x_{a1} + x_{a2} + x_{a3} = 210, x_{b1} + x_{b2} = 120\}$.
We have chosen the starting point $x^0 = (70, 70, 70, 60, 60)$ and we have obtained the solution $x^* = (120, 90, 0, 70, 50)$.
In this case the M-files corresponding to the problem are *useropt.m* and *inputuseropt.m*.

1.5 Harker's Nash-Cournot Problem

In [5], Harker carries this test problem defined as follows.
We assume

N number of firms $i = 1, .., N$;

$x = (x_i)$ production vector $i = 1, .., N$,

firm i produces a quantity x_i of the good;

$Q = \sum x_i$ the total sum of the goods;

$p(Q)$ inverse demand function;

$C_i(x_i)$ the production cost for firm i.

In our example, the functions $C_i(x_i), p(Q)$ are defined as follows:

$$p(Q) = 5000^{\frac{1}{\gamma}} Q^{-\frac{1}{\gamma}}$$

$$C_i(x_i) = c_i x_i + \frac{b_i}{1 + b_i} L_i^{\frac{1}{b_i}} x_i^{\frac{b_i+1}{b_i}}.$$

The function is given by:

$$F_i(x) = C_i^{'}(x_i) - p(Q) - x_i p^{'}(Q);$$

in vectorial form the function can be expressed as:

$$F(x) = \left[c + L^{\frac{1}{b}} x^{\frac{1}{b}} - p(Q)(e - \frac{x}{\gamma Q}) \right]$$

with $c_i, L_i, b_i, \gamma \in \Re^+$ and $\gamma \geq 1$.

We have implemented the example in two cases:

- $N = 5$

  $c = [10, 8, 64, 2]^T$

  $b = [1.2, 1.10, 1, 0.9, 0.8]^T$

  $L = [5, 5, 5, 5, 5]^T$

  $e = [1, 1, 1, 1, 1]^T$

  $\gamma = 1.1$

  If the feasible region is

  $$C = \{x \in \Re^5_+ | x_1 + x_2 + x_3 + x_4 + x_5 = 5\}$$

  we have a VIP.

  Possible starting points are:

$$x^0 = (1, 1, 1, 1, 1),$$

$$x^0 = (10, 10, 10, 10, 10),$$

$$x^0 = (5, 0, 0, 0, 0)$$

and the solution is $x^* = (0.97, 0.99, 1.00, 1.01, 1.01)$.
In this case the M-files corresponding to the problem are *inputHarnashc5.m*, and *Harnashc5.m*.

If the feasible set is

$$C = \{x \in \Re^5 | x \geq 0\}$$

we have an NCP.

Possible starting points are:

$$x^0 = (1, 1, 1, 1, 1),$$

$$x^0 = (10, 10, 10, 10, 10),$$

$$x^0 = (5, 0, 0, 0, 0)$$

and the solution is $x^* = (15.41, 12.50, 9.66, 7.16, 5.13)$.
In this case the M-files corresponding to the problem are: *inputHarnashc5box.m*, and *Harnashc5.m*

- $N = 10$
  $c = [5, 3, 8, 5, 1, 3, 7, 4, 6, 3]^T$
  $b = [1.2, 1, 0.9, 0.6, 1.5, 1, 0.7, 1.1, 0.95, 0.75]^T$
  $L = [10, 10, 10, 10, 10, 10, 10, 10, 10, 10]^T$
  $e = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^T$
  $\gamma = 1.2$

  If the feasible region is

$$C = \{x \in \Re^{10}_+ | \sum_{i=1}^{10} x_i = 10\},$$

we have a VIP.

We may used one of these starting points:

$$x^0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$
$$x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10)$$

and the solution obtained is
$x^* = (1.20, 1.12, 0.83, 0.55, 1.58, 1.12, 0.64, 1.17, 0.95, 0.79)$.

In this case the M-files corresponding to the problem are: *inputHarnashc10.m*, and *Harnashc10.m*.
If the feasible region is

$$C = \{x \in \Re^{10} | x \geq 0\},$$

we have NCP.

We may used one of these starting points:

$$x^0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

$$x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10)$$

and the solution obtained is
$x^* = (7.44, 4.09, 2.59, 0.93, 17.93, 4.09, 1.3, 5.59, 3.22, 1.67)$.
In this case the M-files corresponding to the problem are: *inputHarnashc10box.m*, and *Harnashc10.m*.
When we choice the starting point $x^0 = [10, .., 10]$ we observe that $x^0 \notin C$; then the true starting point is x^1 obtained in the first iteration.
The examples are also used in MCPLIB (see [3]).

1.6 Pang and Murphy's Nash-Cournot Problem

The following test problem is defined in [4] and in [8]. We assume

N number of firms $i = 1, .., N$;

$x = (x_i)$ production vector $i = 1, .., N$,

firm i produces a quantity x_i of the good;

$Q = \sum x_i$ the total sum of the goods;

$p(Q)$ inverse demand function;

$C_i(x_i)$ the production cost for firm i.

In our example the functions $C_i(x_i)$ and $p(Q)$ are defined as follows:

$$p(Q) = 5000^{\frac{1}{\gamma}} Q^{-\frac{1}{\gamma}}$$

$$C_i(x_i) = c_i x_i + \frac{b_i}{1+b_i} L_i^{-\frac{1}{b_i}} x_i^{\frac{b_i+1}{b_i}}.$$

The function is given by

$$F_i(x) = C_i^{'}(x_i) - p(Q) - x_i p^{'}(Q)$$

in vectorial form the function can be expressed as follows

$$F(x) = \left[c + L^{-\frac{1}{b}} x^{\frac{1}{b}} - p(Q)(e - \frac{x}{\gamma Q}) \right],$$

with $c_i, L_i, b_i, \gamma \in \Re^+$ and $\gamma \geq 1$.
We have implemented the example in two cases:

- $N = 5$
  $c = [10, 8, 6, 4, 2]^T$
  $b = [1.2, 1.10, 1, 0.9, 0.8]^T$
  $L = [5, 5, 5, 5, 5]^T$
  $e = [1, 1, 1, 1, 1]^T$
  $\gamma = 1.1$

  If the feasible region is

  $$C = \{x \in \Re_+^5 | x_1 + x_2 + x_3 + x_4 + x_5 = 5\}$$

  we have a VIP.

  Several possible the starting points are:

$$x^0 = (1,1,1,1,1),$$
$$x^0 = (10,10,10,10,10),$$
$$x^0 = (5,0,0,0,0)$$

and a solution is $x^* = (0.95, 0.97, 0.99, 1.02, 1.04)$.
In this case the M-files corresponding to the problem are: *inputPMnashc5.m* and *PMnashc5.m*.

If the feasible set is

$$C = \{x \in \Re^5 | x \geq 0\},$$

we have a NCP.

Several possible the starting points are:

$$x^0 = (1,1,1,1,1),$$
$$x^0 = (10,10,10,10,10),$$
$$x^0 = (5,0,0,0,0)$$

and a solution is $x^* = (36.92, 41.73, 43.68, 42.68, 39.19)$.
In this case the M-files corresponding to the problem are: *inputPMnashc5box.m* and *PMnashc5.m*.

- $N = 10$
  $c = [5,3,8,5,1,3,7,4,6,3]^T$
  $b = [1.2,1,0.9,0.6,1.5,1,0.7,1.1,0.95,0.75]^T$
  $L = [10,10,10,10,10,10,10,10,10,10]^T$
  $e = [1,1,1,1,1,1,1,1,1,1]^T$
  $\gamma = 1.2$

  If the feasible region is

$$C = \{x \in \Re^1 0_+ | \sum_{i=1}^{10} x_i = 10\},$$

we have a VIP.

We may used one of these starting points:

$x^0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$,

$x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10)$

and the solution obtained is
$x^* = (0.96, 1.1, 0.76, 0.97, 1.22, 1.10, 0.83, 1.03, 0.89, 1.10)$.

In this case the M-files corresponding to the problem are: *inputPMnashc10.m* and *PMnashc10.m*.
If the feasible region is

$$C = \{x \in \Re^{10} | x \geq 0\},$$

we have NCP.

We may used one of these starting points:

$x^0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$,

$x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10)$

and the solution obtained is
$x^* = (35.37, 46.57, 4.72, 19.91, 120.93, 46.57, 12, 42.56, 20.59, 32.98)$.
In this case the M-files corresponding to the problem are: *inputPMnashc10box.m* and *PMnashc10.m*.
When we choice the starting point $x^0 = [10, .., 10]$ we observe that $x^0 \notin C$; then the true starting point is x^1 obtained in the first iteration.

1.7 HPHard problem test

In [5], Harker describes a procedure to build an affine function $F(x)$:

$$F(x) = Mx + q,$$

where the matrix M is randomly generated as:

$$M = AA^T + B + D.$$

there any entry of the square nxn matrix A and of the nxn skew-symmetric matrix B is uniformly generated from $(-5, 5)$, and any entry of the diagonal matrix D is uniformly generated from $(0, 0.3)$; consequently, the matrix M is positive definite.
The vector q has been uniformly generated from $(-500, 0)$.

- If the feasible region is

$$C = \{x \in \Re^n_+ | \sum_{i=1}^{n} x_i = n\},$$

  we have a VIP. A possible starting point is $x^0 = (1, ..., 1)$ and the solution.
  We have analyzed two case:

  - **if** $n = 20$, the solution is

$$x^* = (0, 0, 1.71, 3.22, 1.95, 0, 0, 2.37, 0, 1.86,$$

$$1.93, 1.18, 0, 0, 0, 0.39, 1.68, 0.36, 1.44, 1.84)$$

    and the M-files corresponding to the problem are *inputHpHard.m* and *Hphard.m*.
  - **if** $n = 30$, the solution is

$$x^* = (0, 0, 1.13, 2.61, 0, 0.51, 0, 1.31, 2.52, 0.16, 3.43, 1.88, 0, 0, 0.80,$$

$$0, 0.61, 0, 3.36, 2.17, 0, 0, 0, 1.16, 1.09, 2.06, 2.80, 0.79, 0, 1.52)$$

    and the M-files corresponding to the problem are *inputHpHard30.m* and *Hphard.m*.

- If the feasible region is

$$C = \{x \in \Re^n | x \geq 0\},$$

  we have a NCP.

  - **if** $n = 20$, the solution is

$$x^* = (0.09, 1.31, 4.81, 23.31, 1.12, 0, 0, 22.35, 0, 12.60,$$

$$5.50, 6.36, 0, 15.69, 0, 0, 6.16, 12.23, 4.81, 11.94).$$

and the M-files corresponding to the problem are *inputHpHard-box.m* and *Hphard.m*.

- **if** $n = 30$, the solution is

$$x^* = (0, 0, 5.28, 9.84, 0, 2.35, 0.61, 3.83, 11.06, 0, 8.08, 3.71, 0, 0.19, 1.57,$$

$$0, 0.05, 7, 10.95, 6.31, 0.42, 0, 0, 5.42, 2.13, 5.11, 7.35, 2.90, 0, 5.08)$$

and the M-files corresponding to the problem are *inputHpHard30box.m* and *Hphard.m*.

1.8 Obstacle problem

The obstacle problem [1] consist of finding the equilibrium position of an elastic membrane subject to vertical force pushing upward.
The membrane's equilibrium position is its position of minimum energy, where the discretized energy is given by the quadratic function $f(x)$ in the following quadratic problem:

$$\min_{x_l \leq x_i \leq x_u} f(x) \quad = \quad \frac{1}{2} x^T M x^T - q^T x$$

The optimality condition for minimizing the discretized $f(x)$ can be written as following:

$$\begin{aligned}
F(x) - \Pi_l^T \lambda_l + \Pi_u^T \lambda_u &= 0 \\
\lambda_l^T (x_l - l) &= 0 \\
\lambda_u^T (u - x_u) &= 0 \\
\lambda_l \geq 0, \lambda_u &\geq 0
\end{aligned}$$

we can write Mixed Complementarity Problem (MCP):

$$\begin{aligned}
&F_i(x) > 0 \quad and \quad x_l = l \\
&F_i(x) = 0 \quad and \quad x_l < x < x_u \\
&F_i(x) < 0 \quad and \quad x_u = u
\end{aligned}$$

We consider a block diagonal matrix $M \in \Re^{N\times N}$ in which every block is a tridiagonal submatrix $A_i \in \Re^{n\times n}$, with this form:

$$\mathbf{M} = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ...0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & ...0 \\ 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & ...0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & ...0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & ...0 \\ ... \\ ... \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ...A_N \end{bmatrix}.$$

with

$$\mathbf{A_i} = \begin{bmatrix} -4 & -1 & 0 & 0 & 0 & 0 \\ -1 & -4 & -1 & 0 & 0 & 0 \\ 0 & .. & .. & .. & 0 & 0 \\ 0 & 0 & ... & ... & ... & 0 \\ 0 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & 0 & -1 & -4 \end{bmatrix}.$$

We have chosen $n = 6$, $N = n * n$ and $q = -\frac{k}{(N+1)*(N+1)}$ with $k = 1$.
The feasible set is $C = \{x \in \Re^n | x_l \le x_i \le x_u\}$, where x_l, x_u are computed as follows:

do $j = 1 : N$
    do $i = 1 : N$
        $x_l(i + (j-1) * N) =$
$(\sin(9.2 * (i-1)/(N+1.0)) * \sin(9.3 * (j-1)/(N+1.0)))^3$
        $x_u(i + (j-1) * N) =$
$(\sin(9.2 * (i-1)/(N+1.0)) * \sin(9.3 * (j-1)/(N+1.0)))^2 + 0.02$
    enddo
enddo

We have chosen the starting point $x^0 = (0.1, ..., 0.1)$ and we have obtained the solution

$$\begin{aligned} x^* = & [0.0151, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.828, 0.247, 0.1262 \\ & 0.0854, 0.0408, 0.02, 0.2229, 0.0722, 0.1319, 0.1341, 0.0375, 0.02, 0.1187 \\ & 0.1304, 0.1878, 0.2612, 0.0651, 0.02, 0.0811, 0.1223, 0.2075, 0.3503, 0.0749 \\ & 0.02, 0.0428, 0.0428, 0.0839, 0.1106, 0.03] \end{aligned}$$

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