A MatLab collection of variational inequality problems

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Abstract

In order to give a uniform basis for testing several algorithms, in this work, we have collected a set of variational inequality problems. At the URL (http://dm.unife.it/pn2o/software.html) the MatLab M-script and M-function files related to the considered test problems are downloadable (TESTVIPs).

Introduction

We consider the classical variational inequality problem VIP(F,C), which is to find a point $x^*$ such that

$$x^* \in C, \quad <F(x^*), x - x^*> \geq 0 \quad \forall x \in C$$

(1)

where $C$ is a nonempty closed convex subset of $\mathbb{R}^n$, $<.,.>$ the usual inner product in $\mathbb{R}^n$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function. Let $C^*$ be the set of the solutions.

In the special case where $C = \mathbb{R}^*_n$, problem (1) is a nonlinear complementary problem (NCP):

$$x^* \geq 0, \quad F(x^*) \geq 0 \quad \text{and} \quad <x^*, F(x^*)> = 0.$$  

(2)

If $F$ is affine, $F(x) = Mx + q$ where $M \in \mathbb{R}^{nxn}$ (is a positive semidefinite matrix) and $q \in \mathbb{R}^n$, then the problem (1) is an affine variational inequality
problem and (2) is a linear complementary problem (LCP).
In this work we collect a set of test problems arising from the literature. For any problem, we report at least a solution and we propose an initial point to start an iterative method.
Each test problem is related to two files: the first is an M-script file where the constraints and other parameters of the problem are defined. The variable $fs$ is put equal to the string containing the name of the M-function file containing the definition of $F$.
For example, the first test problem (Mathiesen’s problem) is related to the following two files inputmathiesen.m and mathiesen.m:

- %script (inputmathiesen.m)
  %
  %constraint (VIP)
  n=3; %dimension
  A=[1,-1,-1]; %constraint ($A*x<=b$)
  b=0;
  Aeq=[1 1 1]; %equality constraint ($Aeq*x=beq$)
  beq=1;
  lb=[eps,eps,0]'; %lower bound
  ub=[]; %upper bound
  %
  %starting point
  x=[.1 .8 .1]';
  %x=[0.4, 0.3,0.3]';
  %
  %function F
  fs='mathiesen';

- %M-function (mathiesen.m)
  function [f]=mathiesen(x)
  %
  %Reference:
L. Mathiesen,  
'An algorithm based on a sequence of linear 
complementary problems applied to a Walrasian 
equilibrium model: an example',  

f = \begin{bmatrix} 
0.9(5x(2) + 3x(3))/x(1) \\
0.1(5x(2) + 3x(3))/x(2) - 5 \\
-3 
\end{bmatrix}; 

In order to use test problem it is sufficient to insert the following instruction 
in the code implementing a method:

eval('inputmathiesen')

1 Test Problems

1.1 Mathiesen’s Problem

This problem was used first by Mathiesen [7]. The function $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is

$$F(x) = - \begin{bmatrix} 
0.9(5x_2 + 3x_3)/x_1 \\
0.1(5x_2 + 3x_3)/x_2 - 5 \\
-3 
\end{bmatrix},$$

and its feasible set is:

$$C = \{ x \in \mathbb{R}_+^3 | x_1 + x_2 + x_3 = 1, \quad x_1 - x_2 - x_3 \leq 0 \}.$$ 

This example is an Walrasian model in which the consumer demand function is determined by a single consumer; there is one production activity, and three goods.

The M-files corresponding to the problem are: inputmathiesen.m and matiesen.m.

We propose to use as starting points are $x^0 = (0.1, 0.8, 0.1)$ or $x^0 = (0.4, 0.3, 0.3)$.

A solution of the problem is $x^* = (0.5, 0.08, 0.41).$
1.2 Kojima-Shindo’s Problem

In this problem test [4], the function $F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is defined as follows:

$$
F(x) = \begin{bmatrix}
3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\
2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\
x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\
x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3
\end{bmatrix}.
$$

If the feasible set is the following simplex:

$$
C = \{ x \in \mathbb{R}_+^4 \mid x_1 + x_2 + x_3 + x_4 = 4 \},
$$

we have a VIP.

A possible choice of the starting point is $x^0 = (2, 0, 0, 2)$ and, for this example, we can obtain as solution to this problem the point: $x^1 = (1.22, 0, 0, 0.5)$. In this case the two M-files are `inputkojshi.m` and `kojshi.m`.

If the feasible region is the nonnegative orthant of $\mathbb{R}^4$, $x \in \mathbb{R}_+^4$, we have a NCP. In this case a possible choice of the starting point is $x^0 = (2, 0, 0, 2)$ and for this example we can obtain as solution the point: $x^1 = (1.22, 0, 0, 0.50)$. In this case the two M-files are `inputkojshibox.m` and `kojshi.m`.

1.3 Braess Network Problem

In [6], Marcotte considers the Braess paradox network with the separable linear cost function illustrated in Figure 1, in which the arcs are ordered as follows:

$$(1, 2), (1, 3), (2, 3), (2, 4), (3, 4).$$

The delay function is:

$$
F(x) = \begin{bmatrix}
10 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 10
\end{bmatrix} \begin{bmatrix}
x_{12} \\
x_{13} \\
x_{23} \\
x_{24} \\
x_{34}
\end{bmatrix} + \begin{bmatrix}
0 \\
50 \\
10 \\
50 \\
0
\end{bmatrix}.
$$
In this example the feasible set is:

\[ C = \{ x \in \mathbb{R}_+^5 | Bx = b \}, \]

where the node-arc incidence matrix and \( b \) are respectively as follows:

\[
B = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 \\
0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & -1 & -1 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
6 \\
0 \\
0 \\
-6 \\
\end{bmatrix}.
\]

This matrix has rank three; then we have considered the full row-rank \( \hat{B} \) submatrix of \( B \), and the vector \( \hat{b} \), as follows

\[
\hat{B} = \begin{bmatrix}
-1 & 0 & 1 & 1 & 0 \\
0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 \\
\end{bmatrix}, \quad \hat{b} = \begin{bmatrix}
0 \\
0 \\
-6 \\
\end{bmatrix}.
\]

A reasonable choice of the starting point is \( x^0 = (6, 0, 6, 0, 6) \) and the solution is \( x^* = (4, 2, 2, 2, 4) \).

In this case the M-files corresponding to the problem are \texttt{inputbraessnet.m} and \texttt{braessnet.m}.

### 1.4 User-Optimized Traffic Pattern

In [2], Dafermos computes user-optimization traffic pattern for the sim-
ple network shown in Figure 2, with only two nodes $x, y$ and five links $a_1, a_2, a_3, b_1, b_2$, where $a_1, a_2, a_3$ are directed from $x$ to $y$ and $b_1, b_2$ are the return of $a_1, a_2$ respectively.

The travel cost functions are given by

$$F(x) = \begin{bmatrix} 10 & 0 & 0 & 5 & 0 \\ 0 & 15 & 0 & 0 & 5 \\ 0 & 0 & 20 & 0 & 0 \\ 2 & 0 & 0 & 20 & 0 \\ 0 & 1 & 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \\ x_{b1} \\ x_{b2} \end{bmatrix} + \begin{bmatrix} 1000 \\ 950 \\ 3000 \\ 1000 \\ 1300 \end{bmatrix};$$

further the problem is subjected to the following constraints:

$$C = \{x \in \mathbb{R}_+^5 | x_{a1} + x_{a2} + x_{a3} = 210, x_{b1} + x_{b2} = 120\}.$$

We have chosen the starting point $x^0 = (70, 70, 70, 60, 60)$ and we have obtained the solution $x^* = (120, 90, 0, 70, 50)$.

In this case the M-files corresponding to the problem are `useropt.m` and `inputuseropt.m`.

### 1.5 Harker’s Nash-Cournot Problem

In [5], Harker carries this test problem defined as follows.

We assume

$N$ number of firms $i = 1,..,N$;
\(x = (x_i)\) production vector \(i = 1, ..., N\),

firm \(i\) produces a quantity \(x_i\) of the good;

\(Q = \sum x_i\) the total sum of the goods;

\(p(Q)\) inverse demand function;

\(C_i(x_i)\) the production cost for firm \(i\).

In our example, the functions \(C_i(x_i), p(Q)\) are defined as follows:

\[ p(Q) = 5000^{\frac{1}{\gamma}}Q^{-\frac{1}{\gamma}} \]

\[ C_i(x_i) = c_i x_i + \frac{b_i}{1 + b_i} L_i b_i x_i^{b_i+1} \]

The function is given by:

\[ F_i(x) = C_i'(x_i) - p(Q) - x_i p'(Q); \]

in vectorial form the function can be expressed as:

\[ F(x) = \left[ c + L^T x^T - p(Q)(e - \frac{x}{\gamma Q}) \right] \]

with \(c, L, b, \gamma \in \mathbb{R}^+\) and \(\gamma \geq 1\).

We have implemented the example in two cases:

\[ N = 5 \]

\[ c = [10, 8, 64, 2]^T \]

\[ b = [1.2, 1.10, 1, 0.9, 0.8]^T \]

\[ L = [5, 5, 5, 5]^T \]

\[ e = [1, 1, 1, 1, 1]^T \]

\[ \gamma = 1.1 \]

If the feasible region is

\[ C = \{ x \in \mathbb{R}_+^5 | x_1 + x_2 + x_3 + x_4 + x_5 = 5 \} \]

we have a VIP.

Possible starting points are:
\[ x^0 = (1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10), \]
\[ x^0 = (5, 0, 0, 0, 0) \]

and the solution is \( x^* = (0.97, 0.99, 1.00, 1.01, 1.01) \).

In this case the M-files corresponding to the problem are \textit{inputHarnasch5.m}, and \textit{Harnasch5.m}.

If the feasible set is \[ C = \{ x \in \mathbb{R}^5 | x \geq 0 \} \]
we have an NCP.

Possible starting points are:
\[ x^0 = (1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10), \]
\[ x^0 = (5, 0, 0, 0, 0) \]

and the solution is \( x^* = (15.41, 12.50, 9.66, 7.16, 5.13) \).

In this case the M-files corresponding to the problem are: \textit{inputHarnasch5box.m}, and \textit{Harnasch5.m}.

- \( N = 10 \)
  \[ c = [5, 3, 8, 5, 1, 3, 7, 4, 6, 3]^T \]
  \[ b = [1.2, 1, 0.9, 0.6, 1.5, 1, 0.7, 1.1, 0.95, 0.75]^T \]
  \[ L = [10, 10, 10, 10, 10, 10, 10, 10, 10, 10]^T \]
  \[ e = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^T \]
  \[ \gamma = 1.2 \]

If the feasible region is
\[ C = \{ x \in \mathbb{R}^{10}_+ | \sum_{i=1}^{10} x_i = 10 \}, \]
we have a VIP.

We may used one of these starting points:

\[ x^0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10) \]

and the solution obtained is

\[ x^* = (1.20, 1.12, 0.83, 0.55, 1.58, 1.12, 0.64, 1.17, 0.95, 0.79). \]

In this case the M-files corresponding to the problem are: `inputHarnashc10.m`, and `Harnashc10.m`.

If the feasible region is

\[ C = \{ x \in \mathbb{R}^{10} | x \geq 0 \}, \]

we have NCP.

We may used one of these starting points:

\[ x^0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10) \]

and the solution obtained is

\[ x^* = (7.44, 4.09, 2.59, 0.93, 17.93, 4.09, 1.3, 5.59, 3.22, 1.67). \]

In this case the M-files corresponding to the problem are: `inputHarnashc10box.m`, and `Harnashc10.m`.

When we choice the starting point \( x^0 = [10, \ldots, 10] \) we observe that \( x^0 \notin C \); then the true starting point is \( x^1 \) obtained in the first iteration.

The examples are also used in MCPLIB (see [3]).

### 1.6 Pang and Murphy’s Nash-Cournot Problem

The following test problem is defined in [4] and in [8]. We assume

\[ N \] number of firms \( i = 1, \ldots, N; \]
\( x = (x_i) \) production vector \( i = 1, \ldots, N \),

firm \( i \) produces a quantity \( x_i \) of the good;

\( Q = \sum x_i \) the total sum of the goods;

\( p(Q) \) inverse demand function;

\( C_i(x_i) \) the production cost for firm \( i \).

In our example the functions \( C_i(x_i) \) and \( p(Q) \) are defined as follows:

\[
p(Q) = 5000 \gamma^\frac{1}{\gamma} Q^{-\frac{1}{\gamma}}
\]

\[
C_i(x_i) = c_i x_i + \frac{b_i}{1+b_i} L_i^{-\frac{1}{b_i}} x_i^\frac{b_i+1}{b_i}.
\]

The function is given by

\[
F_i(x) = C_i'(x_i) - p(Q) - x_i p'(Q)
\]

in vectorial form the function can be expressed as follows

\[
F(x) = \left[ c + L^{\frac{1}{\gamma}} x^\frac{1}{\gamma} - p(Q)(e - \frac{x}{\gamma Q}) \right],
\]

with \( c_i, L_i, b_i, \gamma \in \mathbb{R}^+ \) and \( \gamma \geq 1 \).

We have implemented the example in two cases:

- \( N = 5 \)
  
  \[
  c = [10, 8, 6, 4, 2]^T
  \]
  
  \[
  b = [1.2, 1.1, 1.1, 0.9, 0.8]^T
  \]
  
  \[
  L = [5, 5, 5, 5]^T
  \]
  
  \[
  e = [1, 1, 1, 1, 1]^T
  \]
  
  \[
  \gamma = 1.1
  \]

If the feasible region is

\[
C = \{ x \in \mathbb{R}_5^5 | x_1 + x_2 + x_3 + x_4 + x_5 = 5 \}
\]

we have a VIP.

Several possible the starting points are:
\[ x^0 = (1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10), \]
\[ x^0 = (5, 0, 0, 0, 0) \]

and a solution is \( x^* = (0.95, 0.97, 0.99, 1.02, 1.04) \).

In this case the M-files corresponding to the problem are: \textit{inputPMnashc5.m} and \textit{PMnashc5.m}.

If the feasible set is
\[ C = \{ x \in \mathbb{R}^5 | x \geq 0 \}, \]
we have a NCP.

Several possible the starting points are:
\[ x^0 = (1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10), \]
\[ x^0 = (5, 0, 0, 0, 0) \]

and a solution is \( x^* = (36.92, 41.73, 43.68, 42.68, 39.19) \).

In this case the M-files corresponding to the problem are: \textit{inputPMnashc5box.m} and \textit{PMnashc5.m}.

- \( N = 10 \)
\[ c = [5, 3, 8, 5, 1, 3, 7, 4, 6, 3]^T \]
\[ b = [1.2, 1, 0.9, 0.6, 1.5, 1, 0.7, 1.1, 1.1, 0.95, 0.75]^T \]
\[ L = [10, 10, 10, 10, 10, 10, 10, 10, 10, 10]^T \]
\[ e = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^T \]
\[ \gamma = 1.2 \]

If the feasible region is
\[ C = \{ x \in \mathbb{R}^{10}_+ | \sum_{i=1}^{10} x_i = 10 \}, \]
we have a VIP.

We may used one of these starting points:

\[ x^0 = (1, 1, 1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10) \]

and the solution obtained is
\[ x^* = (0.96, 1.1, 0.76, 0.97, 1.22, 1.10, 0.83, 1.03, 0.89, 1.10) \]

In this case the M-files corresponding to the problem are: `inputPMnashe10.m` and `PMnashe10.m`.

If the feasible region is
\[ C = \{ x \in \mathbb{R}^{10} | x \geq 0 \} \]
we have NCP.

We may used one of these starting points:

\[ x^0 = (1, 1, 1, 1, 1, 1, 1, 1), \]
\[ x^0 = (10, 10, 10, 10, 10, 10, 10, 10, 10) \]

and the solution obtained is
\[ x^* = (35.37, 46.57, 4.72, 19.91, 120.93, 46.57, 12, 42.56, 20.59, 32.98) \]

In this case the M-files corresponding to the problem are: `inputPMnashe10box.m` and `PMnashe10.m`.

When we choice the starting point \( x^0 = [10, \ldots, 10] \) we observe that \( x^0 \notin C \); then the true starting point is \( x^1 \) obtained in the first iteration.

### 1.7 HPHard problem test

In [5], Harker describes a procedure to build an affine function \( F(x) \):

\[ F(x) = Mx + q, \]

where the matrix \( M \) is randomly generated as:

\[ M = AA^T + B + D. \]
there any entry of the square $nxn$ matrix $A$ and of the $nxn$ skew-symmetric matrix $B$ is uniformly generated from $(-5, 5)$, and any entry of the diagonal matrix $D$ is uniformly generated from $(0, 0.3)$; consequently, the matrix $M$ is positive definite.

The vector $q$ has been uniformly generated from $(-500, 0)$.

- If the feasible region is
  \[ C = \{ x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = n \}, \]
  we have a VIP. A possible starting point is $x^0 = (1, ..., 1)$ and the solution.
  We have analyzed two case:
  - if $n = 20$, the solution is
    \[
    x^* = (0, 0, 1.71, 3.22, 1.95, 0, 0, 2.37, 0, 1.86, \\
    1.93, 1.18, 0, 0, 0.39, 1.68, 0.36, 1.44, 1.84)
    \]
    and the M-files corresponding to the problem are inputHpHard.m and Hphard.m.
  - if $n = 30$, the solution is
    \[
    x^* = (0, 0, 1.13, 2.61, 0, 0.51, 0, 1.31, 2.52, 0.16, 3.43, 1.88, 0, 0, 0.80, \\
    0, 0.61, 0, 3.36, 2.17, 0, 0, 0, 1.16, 1.09, 2.06, 2.80, 0.79, 0, 1.52)
    \]
    and the M-files corresponding to the problem are inputHpHard30.m and Hphard.m.

- If the feasible region is
  \[ C = \{ x \in \mathbb{R}^n | x \geq 0 \}, \]
  we have a NCP.
  - if $n = 20$, the solution is
    \[
    x^* = (0.09, 1.31, 4.81, 23.31, 1.12, 0, 0, 22.35, 0, 12.60,
    \]

and the M-files corresponding to the problem are `inputHpHardbox.m` and `Hphard.m`.

- if $n = 30$, the solution is

$$x^* = (0, 0, 5.28, 9.84, 0, 2.35, 0.61, 3.83, 11.06, 0, 8.08, 3.71, 0, 0.19, 1.57,$$

$$0, 0.05, 7, 10.95, 6.31, 0.42, 0, 0, 5.42, 2.13, 5.11, 7.35, 2.90, 0, 5.08)$$

and the M-files corresponding to the problem are `inputHpHard30box.m` and `Hphard.m`.

### 1.8 Obstacle problem

The obstacle problem [1] consists of finding the equilibrium position of an elastic membrane subject to vertical force pushing upward. The membrane’s equilibrium position is its position of minimum energy, where the discretized energy is given by the quadratic function $f(x)$ in the following quadratic problem:

$$\min_{x_l \leq x_i \leq x_u} f(x) = \frac{1}{2} x^T M x^T - q^T x$$

The optimality condition for minimizing the discretized $f(x)$ can be written as following:

$$F(x) - \Pi_l^T \lambda_l + \Pi_u^T \lambda_u = 0$$

$$\lambda_l^T (x_l - l) = 0$$

$$\lambda_u^T (u - x_u) = 0$$

$$\lambda_l \geq 0, \lambda_u \geq 0$$

we can write Mixed Complementarity Problem (MCP):

$$F_i(x) > 0 \quad \text{and} \quad x_l = l$$

$$F_i(x) = 0 \quad \text{and} \quad x_l < x < x_u$$

$$F_i(x) < 0 \quad \text{and} \quad x_u = u$$
We consider a block diagonal matrix $M \in \mathbb{R}^{N \times N}$ in which every block is a tridiagonal submatrix $A_i \in \mathbb{R}^{n \times n}$, with this form:

$$M = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots A_N \end{bmatrix}.$$

with

$$A_1 = \begin{bmatrix} -4 & -1 & 0 & 0 & 0 \\ -1 & -4 & -1 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 \end{bmatrix}.$$

We have chosen $n = 6$, $N = n \times n$ and $q = -\frac{k}{(N+1)^2(N+1)}$ with $k = 1$.

The feasible set is $C = \{x \in \mathbb{R}^n | x_l \leq x_i \leq x_u \}$, where $x_l, x_u$ are computed as follows:

\[
\begin{align*}
\text{do } & j = 1 : N \\
& \text{ do } i = 1 : N \\
& \quad x_l(i + (j - 1) \times N) = (\sin(9.2 \times (i - 1)/(N + 1))) \times \sin(9.3 \times (j - 1)/(N + 1)))^3 \\
& \quad x_u(i + (j - 1) \times N) = (\sin(9.2 \times (i - 1)/(N + 1))) \times \sin(9.3 \times (j - 1)/(N + 1)))^2 + 0.02 \\
& \quad \text{ enddo} \\
\text{ enddo}
\end{align*}
\]

We have chosen the starting point $x^0 = (0.1, \ldots, 0.1)$ and we have obtained
the solution

\[ x^* = [0.0151, 0.02, 0.02, 0.02, 0.02, 0.02, 0.02, 0.828, 0.247, 0.1262 \\
0.0854, 0.0408, 0.02, 0.2229, 0.0722, 0.1319, 0.1341, 0.0375, 0.02, 0.1187 \\
0.1304, 0.1878, 0.2612, 0.0651, 0.02, 0.0811, 0.1223, 0.2075, 0.3503, 0.0749 \\
0.02, 0.0428, 0.0428, 0.0839, 0.1106, 0.03] \]

References


