A Novel Gradient Projection Approach for Fourier-Based Image Restoration

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Abstract. This work deals with the ill-posed inverse problem of reconstructing a two-dimensional image of an unknown object starting from sparse and nonuniform measurements of its Fourier Transform. In particular, if we consider a priori information about the target image (e.g., the nonnegativity of the pixels), this inverse problem can be reformulated as a constrained optimization problem, in which the stationary points of the objective function can be viewed as the solutions of a deconvolution problem with a suitable kernel. We propose a fast and effective gradient-projection iterative algorithm to provide regularized solutions of such a deconvolution problem by early stopping the iterations. Preliminary results on a real-world application in astronomy are presented.

Keywords: inverse problems, image reconstruction, Fourier Transform, gradient projection methods, solar flares
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INTRODUCTION

The reconstruction of an image from the knowledge of a nonuniform sampling of its Fourier Transform is a common problem in several scientific areas, such as radioastronomy, X-ray astronomy, computed tomography and magnetic resonance imaging [5, 7, 8, 12]. In the general case, given N points irregularly distributed in the frequency space, we want to recover the corresponding image in the physical space. This problem presents both theoretical and computational difficulties: from the theoretical point of view, we are dealing with an inverse problem which is ill posed due to both the sparsity of the samples in the frequency space and the noise affecting the data. From the computational point of view, the irregularity of the data sampling prevents the straightforward application of the Fast Fourier Transform (FFT) algorithm, with the result of a heaviness of the inversion procedure.
Several methods have been proposed in the recent literature to face these difficulties (see [20] and references therein), most of which are based on a *gridding* step followed by a FFT-based iterative procedure. In the present paper we introduce a new method (that we called *Space-D*) to reconstruct an image from a nonuniform sampling of its Fourier Transform which acts straightly on the data without interpolation and re-sampling operations, exploiting in this way the real nature of the data themselves. In particular, our target image is a solution of an appropriate deconvolution problem and its reconstruction is effectively computed by means of a gradient projection method with an adaptive steplength parameter, which is chosen via a suitable alternation of the two Barzilai-Borwein rules [1].
Since it has been experimentally shown that the gradient projection method exhibits a semi-convergence behaviour [2, 6], regularized solutions can be achieved by early stopping the iterations. Moreover, super-resolution is obtained through projections of the images obtained at each iteration of the algorithm on the admissible region [17].
As a real-world playground for the numerical experimentation of our algorithm we chose the spatial reconstruction of X-ray emission during the solar flares through the analysis of datasets provided by the Reuven Ramaty High Energy Solar Spectroscopic Imager (RHESSI) spacecraft [15].

FORMULATION OF THE PROBLEM

Let us consider a uniform grid over the square $\mathscr{D} = [X_1, X_2] \times [Y_1, Y_2]$ in the image plane given by the points $x_j = X_1 + (j-1)\Delta x,\ y_h = Y_1 + (h-1)\Delta y\ (j,h = 1,...,n)$. If we denote with $\bar{f} \in L^2(\mathscr{D}, \mathbb{C})$ the continuous two-dimensional target function, we can say that our final goal is to find a stable and meaningful approximation of the values $f_{jh} = \bar{f}(x_j, y_h)\Delta x \Delta y\ (j,h = 1,...,n)$. The vector of measured data $g \in \mathbb{C}^N$ (also called *visibilities* [19]) is

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related to the target image $f = \{f_{jh}\}_{j,h=1,\ldots,n} \in \mathbb{C}^{n^2}$ by a rectangular-rule-discretized version of the Fourier Transform: $g_k \approx \sum_{j,h=1}^{n} f_{jh} e^{2\pi i (u_k x_j + v_k y_h)}$ $(k = 1,\ldots,N)$, where the points (u_k, v_k) $(k = 1,\ldots,N)$ represent the available sampling of the frequency plane. Given $f \in \mathbb{C}^{n^2}$ and $c \in \mathbb{C}^N$, let us define the linear operator A and its adjoint A^* as

$$(Af)_k = \sum_{j,h=1}^{n} f_{jh} e^{2\pi i (u_k x_j + v_k y_h)}, \;\; k = 1,\ldots,N; \quad (A^* c)_{jh} = \sum_{k=1}^{N} c_k e^{-2\pi i (u_k x_j + v_k y_h)}, \quad j,h = 1,\ldots,n. \tag{1}$$

The image reconstruction problem can be formulated as a linear inverse problem $Af = g$, which is ill posed since it has infinite solutions. The minimum norm solution (or *generalized solution*) of such an inverse problem is a minimum point of the least squares functional

$$J(f) \equiv \frac{1}{2} \|Af - g\|^2_{\mathbb{C}^N}, \quad f \in \mathbb{C}^{n^2}. \tag{2}$$

From the definition of A and A^* we can find the stationary points of (2) by solving the deconvolution problem $D * f = f^d$, where the *dirty beam* $D = \{D_{jh}\}_{j,h=-n,\ldots,n}$ and the *dirty image* $f^d = \{f^d_{jh}\}_{j,h=1,\ldots,n}$ are defined as [14]

$$D_{jh} = \sum_{k=1}^{N} e^{-2\pi i (u_k x_j + v_k y_h)}, \quad j,h = -n,\ldots,n \quad ; \quad f^d_{jh} = \sum_{k=1}^{N} g_k e^{-2\pi i (u_k x_j + v_k y_h)}, \quad j,h = 1,\ldots,n. \tag{3}$$

Since it is natural in image restoration to require the radiation flux to be real and nonnegative, we are interested in solving the constrained minimization problem

$$\min_{\substack{f \in \mathbb{R}^{n^2} \\ f \geq 0}} \quad J(f). \tag{4}$$

THE SPACE-D ALGORITHM

For the numerical solution of the minimization problem (4) we propose to employ a gradient projection algorithm [3], which is well suited in case of simple constraints. In particular, we adopt the variable steplength approach [4, 6], which applies to any minimization problem involving a differentiable real valued objective function and a convex closed feasible region $\mathscr{C} \subset \mathbb{R}^m$, $m \in \mathbb{Z}_+$. The method starts from a feasible point $f^{(0)} \in \mathscr{C}$ and generates a sequence

$$f^{(k+1)} = f^{(k)} + \lambda_k d^{(k)}, \quad k = 0,1,2,\ldots$$

where each $f^{(k)} \in \mathscr{C}$ and the search direction is computed as

$$d^{(k)} = \mathscr{P}_{\mathscr{C}}(f^{(k)} - \alpha_k \nabla J(f^{(k)})) - f^{(k)}, \quad k = 0,1,2,\ldots.$$

Here, λ_k, α_k are positive scalar parameters and $\mathscr{P}_{\mathscr{C}}$ denotes the orthogonal projection on the set $\mathscr{C}$. If α_k is chosen in a bounded interval and λ_k satisfies an Armijo-type condition, then the limit points of the sequence $\{f^{(k)}\}$ are stationary for problem (4) [4, 6]. In spite of its simplicity, the gradient projection method offers good practical performances on large scale problems compared to other optimization algorithms, especially when the hessian matrix of the objective function is not explicitly available. A crucial feature for the effectiveness of the method is the choice of the steplength parameter α_k. Recent studies show that significant improvements in the convergence speed of gradient methods can be obtained with a suitable alternation of the two Barzilai and Borwein (BB) rules [1], which are defined as follows

$$\alpha_k^{(1)} = \frac{\left(f^{(k)} - f^{(k-1)}\right)^T \left(f^{(k)} - f^{(k-1)}\right)}{\left(f^{(k)} - f^{(k-1)}\right)^T \left(\nabla J(f^{(k)}) - \nabla J(f^{(k-1)})\right)} \quad ; \quad \alpha_k^{(2)} = \frac{\left(f^{(k)} - f^{(k-1)}\right)^T \left(\nabla J(f^{(k)}) - \nabla J(f^{(k-1)})\right)}{\left(\nabla J(f^{(k)}) - \nabla J(f^{(k-1)})\right)^T \left(\nabla J(f^{(k)}) - \nabla J(f^{(k-1)})\right)}.$$

The BB rules are motivated by the quasi-Newton approach, where the inverse of the hessian is replaced by a multiple of the identity matrix $B(\alpha) = \alpha I$. Then, omitting the iteration number, the two BB formulas are given by $\alpha^{(1)} = \arg\min \|B(\alpha)s - y\|$ and $\alpha^{(2)} = \arg\min \|s - B(\alpha)^{-1} y\|$.

The recent literature on the steplength selection in gradient methods suggests to design steplength updating strategies by alternating the two BB rules. Here we will use the adaptive alternation strategy proposed in [11], that recently obtained very good results in signal and image denoising and deblurring problems [2, 6, 21]. Given an initial value α_0, the steplengths α_k ($k = 1,2,\dots$) are defined by the following criterion:

IF $\alpha_k^{(2)}/\alpha_k^{(1)} \le \tau_k$ THEN
$\alpha_k = \min\left\{\alpha_j^{(2)},\ j = \max\{1, k - M_\alpha\},\dots,k\right\}; \qquad \tau_{k+1} = \tau_k * 0.9;$
ELSE
$\alpha_k = \alpha_k^{(1)}; \qquad \tau_{k+1} = \tau_k * 1.1;$
ENDIF

where M_α is a prefixed nonnegative integer and $\tau_1 \in (0,1)$. As observed in several experimental studies [2, 6], the GP method equipped with this alternation of the two BB formulas is more efficient with respect to the same algorithm with other steplength selection rules and also to other iterative approaches as the projected Landweber method.
The stopping criterion for the iterative regularization method in the Space-D algorithm is based on the Morozov's discrepancy principle [10]. In particular, we assume that the measured data g^δ can be represented in the form $g^\delta = Af + \delta g$, where δg denotes the noise and f is the object to be reconstructed. Furthermore, we assume that the quantity $\eta = \|\delta g\|$ (or an estimate of it, as in the case of the application described in the following section) is known. Then, a regularized solution $f^{(k)}$ is computed if we terminate the optimization procedure at the first iteration $\bar{k}$ in which the inequality $\|Af^{(k)} - g^\delta\| \le \eta$ is satisfied.

NUMERICAL EXPERIMENTS: THE RHESSI MISSION

In this section we test our algorithm in a real-world application arising in X-ray astronomy. The specific problem we are interested in is to reconstruct the spatial distribution of the X-ray radiation emitted by accelerated particles in the solar chromosphere during the solar flares. The datasets are provided by the RHESSI satellite, launched by NASA in 2002 with the precise intent to investigate the particle acceleration mechanisms during the solar flares through the spatial and spectral analysis of the emitted X-ray radiation [15]. Because of the RHESSI instruments design, imaging information is recorded as a set of visibilities (varying from tens to two or three hundreds according to the strength of the signal), measured at spatial frequencies (u,v) arranged around concentric circles in the Fourier plane [13].
Following the idea introduced in [16], we created a set of simulated images starting from real solar flare maps. To this aim, we considered one of the most famous events since RHESSI is in orbit, which occurred on July 23 2002 [9, 18] and, starting straightly from the counts collected by RHESSI, we used the Clean algorithm [13] to build the X-ray images corresponding to three different energy ranges: 20–22 keV, 41–46 keV and 156–177 keV. These 64×64 images (with 1 arcsec $\times$ 1 arcsec pixel size) represent very different source morphologies (see figure 1):

- in the low energies case, a single wide asymmetric X-ray source is present (figure 1(a));
- for intermediate energies, a more complicated structure combining the superposition of several sources is involved (figure 1(b));
- at higher energies, the emission is limited to two compact sources plus a weaker one in-between (figure 1(c)).

A threshold-based filter was applied to the Clean images in order to eliminate artefacts potentially introduced by the Clean algorithm. The resulting maps have been considered as the target distributions for the reconstruction algorithms; the corresponding visibilities have been calculated through numerical integration of the Fourier transform and corrupted by realistic noise with the routines *hsi_vis_map2vis.pro* and *hsi_vis_randomize.pro* available within the official Solar SoftWare (SSW) of the RHESSI mission. The code of the algorithm have been written in Interactive Data Language (IDL) and ran on a PC equipped with a 1.66GHz Intel Core Duo T5500. A constant image with total flux equal to $\max_k |g_k|$ as the starting point $f^{(0)}$. The optimal numbers of iterations provided by the stopping criterium have always been between 50 and 80.
In figure 1 we collect (from left to right, respectively) the original sources, the corresponding dirty images and the reconstructed maps for the three simulated cases obtained with and Space-D. The aim of our experimentation is to evaluate the accuracy of the regularized solutions: we just point out that the overall computation was performed in few seconds, which is negligible with respect to the time needed for loading the data. From figure 1 we can observe that, in general, Space-D reveals a good capability both to recover simple geometries and to detect some details present in the original image. We have to admit that extremely complicated morphologies as the one in simulation 2 represent an hard task also for our methodology, even if on the whole the general structure is well reconstructed.

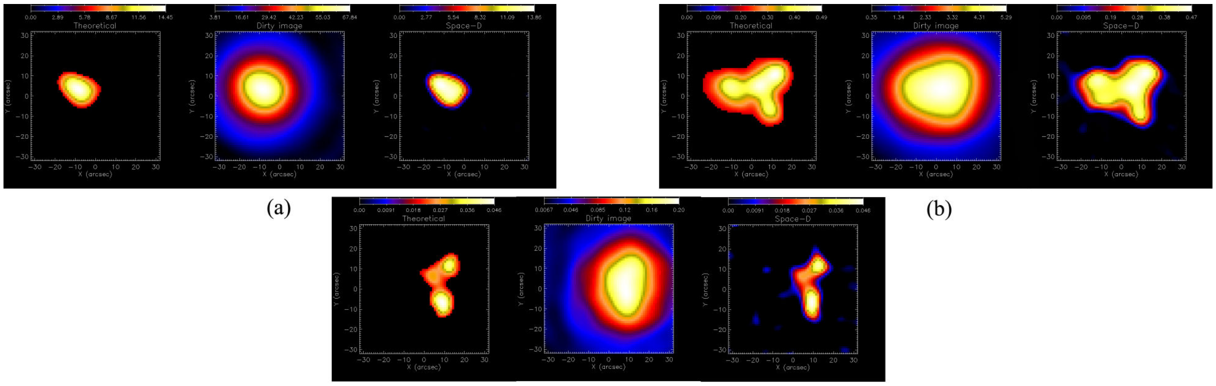

FIGURE 1. Results of the simulated tests for the energy ranges 20-22 keV (a), 41-46 keV (b) and 156-177 keV (c). From left to right, the theoretical image, the dirty image and the reconstruction with Space-D are presented, respectively.

CONCLUSIONS AND FUTURE WORK

In this paper we presented a new deconvolution approach to recover an image from a sparse and nonuniform sampling of its Fourier transform. Applications of this method are possible in image and signal restoration problems arising from several scientific and medical areas. We showed the potential of our algorithm when applied to analyse real and synthetic data from the NASA RHESSI mission. From the numerical experience, the proposed methodology seems to provide very accurate reconstructions in very few seconds, thanks to the well–tried optimization algorithm adopted for the deconvolution.

Future studies will be addressed in both theoretical and practical issues. We could consider further formulations of the problem including more a priori information as, for example, a suitable weighting of the measured visibilities. Moreover, it will be interesting to analyse the behaviour of the algorithm when the regularization is performed by including an explicit penalty term in the objective function (e.g., Tikhonov, entropy, Total Variation).

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