RANK TWO GLOBALLY GENERATED VECTOR BUNDLES
WITH $c_1 \leq 5$.

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Abstract. We classify globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 3$, with $c_1 \leq 5$. The classification is complete but for one case ($n = 3, c_1 = 5, c_2 = 12$).

Introduction.

Vector bundles generated by global sections are basic objects in projective algebraic geometry. Globally generated line bundles correspond to morphisms to a projective space, more generally higher rank bundles correspond to morphism to (higher) Grassmann varieties. For this last point of view (that won’t be touched in this paper) see [11] [12], [13]. Also globally generated vector bundles appear in a variety of problems ([7] just to make a single, recent example).

In this paper we classify globally generated rank two vector bundles on $\mathbb{P}^n$ (projective space over $k$, $\overline{k} = k$, $ch(k) = 0$), $n \geq 3$, with $c_1 \leq 5$. The result is:

**Theorem 0.1.** Let $E$ be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, generated by global sections with Chern classes $c_1, c_2$, $c_1 \leq 5$.

1. If $n \geq 4$, then $E$ is the direct sum of two line bundles.
2. If $n = 3$ and $E$ is indecomposable, then

$$(c_1, c_2) \in S = \{(2, 2), (4, 5), (4, 6), (4, 7), (4, 8), (5, 8), (5, 10), (5, 12)\}.$$

If $E$ exists there is an exact sequence:

$$0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{I}_C(c_1) \rightarrow 0 \quad (*)$$

where $C \subset \mathbb{P}^3$ is a smooth curve of degree $c_2$ with $\omega_C(4 - c_1) \simeq \mathcal{O}_C$. The curve $C$ is irreducible, except maybe if $(c_1, c_2) = (4, 8)$: in this case $C$ can be either irreducible or the disjoint union of two smooth conics.

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(3) For every \((c_1, c_2) \in S, (c_1, c_2) \neq (5, 12)\), there exists a rank two vector bundle on \(\mathbb{P}^3\) with Chern classes \((c_1, c_2)\) which is globally generated (and with an exact sequence as in (2)).

The classification is complete, but for one case: we are unable to say if there exist or not globally generated rank two vector bundles with Chern classes \(c_1 = 5, c_2 = 12\) on \(\mathbb{P}^3\).

1. Rank two vector bundles on \(\mathbb{P}^3\).

1.1. General facts.

For completeness let’s recall the following well known results:

**Lemma 1.1.** Let \(E\) be a rank \(r\) vector bundle on \(\mathbb{P}^n, n \geq 3\). Assume \(E\) is generated by global sections.

1. If \(c_1(E) = 0\), then \(E \simeq r.\mathcal{O}\)
2. If \(c_1(E) = 1\), then \(E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}\) or \(E \simeq T(-1) \oplus (r-n).\mathcal{O}\).

**Proof.** If \(L \subset \mathbb{P}^n\) is a line then \(E|L \simeq \bigoplus_{i=1}^{r} \mathcal{O}_L(a_i)\) by a well known theorem and \(a_i \geq 0, \forall i\) since \(E\) is globally generated. It turns out that in both cases: \(E|L \simeq \mathcal{O}_L(c_1) \oplus (r-1).\mathcal{O}_L\) for every line \(L\), i.e. \(E\) is uniform. Then (1) follows from a result of Van de Ven ([14]), while (2) follows from IV. Prop. 2.2 of [4]. \(\square\)

**Lemma 1.2.** Let \(E\) be a rank two vector bundle on \(\mathbb{P}^n, n \geq 3\). If \(E\) has a nowhere vanishing section then \(E\) splits. If \(E\) is generated by global sections and doesn’t split then \(h^0(E) \geq 3\) and a general section of \(E\) vanishes along a smooth curve, \(C\), of degree \(c_2(E)\) such that \(\omega_C(4 - c_1) \simeq \mathcal{O}_C\). Moreover \(\mathcal{I}_C(c_1)\) is generated by global sections.

**Lemma 1.3.** Let \(E\) be a non split rank two vector bundle on \(\mathbb{P}^3\) with \(c_1 = 2\). If \(E\) is generated by global sections then \(E\) is a null-correlation bundle.

**Proof.** We have an exact sequence: \(0 \to \mathcal{O} \to E \to \mathcal{I}_C(2) \to 0\), where \(C\) is a smooth curve with \(\omega_C(2) \simeq \mathcal{O}_C\). It follows that \(C\) is a disjoint union of lines. Since \(h^0(\mathcal{I}_C(2)) \geq 2\), \(d(C) \leq 2\). Finally \(d(C) = 2\) because \(E\) doesn’t split. \(\square\)

This settles the classification of rank two globally generated vector bundles with \(c_1(E) \leq 2\) on \(\mathbb{P}^3\).
1.2. Globally generated rank two vector bundles with $c_1 = 3$.

The following result has been proved in [13] (with a different and longer proof).

**Proposition 1.4.** Let $E$ be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 3$ then $E$ splits.

**Proof.** Assume a general section vanishes in codimension two, then it vanishes along a smooth curve $C$ such that $\mathcal{O}_C \simeq \mathcal{O}_C(-1)$. Moreover $\mathcal{I}_C(3)$ is generated by global sections. We have $C = \cup_{i=1}^c C_i$ (disjoint union) where each $C_i$ is smooth irreducible with $\mathcal{O}_{C_i} \simeq \mathcal{O}_{C_i}(-1)$. It follows that each $C_i$ is a smooth conic. If $r \geq 2$ let $L = \langle C_1 \rangle \cap \langle C_2 \rangle$ (where $\langle C_i \rangle$ is the plane spanned by $C_i$). Every cubic containing $C$ contains $L$ (because it contains the four points $C_1 \cap L$, $C_2 \cap L$). This contradicts the fact that $\mathcal{I}_C(3)$ is globally generated. Hence $r = 1$ and $E = \mathcal{O}(1) \oplus \mathcal{O}(2)$. \hfill $\square$

1.3. Globally generated rank two vector bundles with $c_1 = 4$.

Let’s start with a general result:

**Lemma 1.5.** Let $E$ be a non split rank two vector bundle on $\mathbb{P}^3$ with Chern classes $c_1,c_2$. If $E$ is globally generated and if $c_1 \geq 4$ then:

$$c_2 \leq \frac{2c_1^3 - 4c_1^2 + 2}{3c_1 - 4}.$$ 

**Proof.** By our assumptions a general section of $E$ vanishes along a smooth curve, $C$, such that $\mathcal{I}_C(c_1)$ is generated by global sections. Let $U$ be the complete intersections of two general surfaces containing $C$. Then $U$ links $C$ to a smooth curve, $Y$. We have $Y \neq \emptyset$ since $E$ doesn’t split. The exact sequence of liaison: $0 \to \mathcal{I}_U(c_1) \to \mathcal{I}_C(c_1) \to \mathcal{O}_Y(4 - c_1) \to 0$ shows that $\mathcal{O}_Y(4 - c_1)$ is generated by global sections. Hence $\deg(\mathcal{O}_Y(4 - c_1)) \geq 0$. We have $\deg(\mathcal{O}_Y(4 - c_1)) = 2g' - 2 + d'(4 - c_1)$ ($g' = p_a(Y)$, $d' = \deg(Y)$). So $g' \geq \frac{d'(4 - c_1) + 2}{2} \geq 0$ (because $c_1 \geq 4$). On the other hand, always by liaison, we have: $g' - g = \frac{1}{2}(d' - d)(2c_1 - 4)$ ($g = p_a(C)$, $d = \deg(C)$). Since $d' = c_1^2 - d$ and $g = \frac{d(4 - c_1)}{2} + 1$ (because $\mathcal{O}_C(4 - c_1) \simeq \mathcal{O}_C$), we get: $g' = 1 + \frac{d(4 - c_1)}{2} + \frac{1}{2}(c_1^2 - 2d)(2c_1 - 4) \geq 0$ and the result follows. \hfill $\square$

Now we have:

**Proposition 1.6.** Let $E$ be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 4$ and if $E$ doesn’t split, then $5 \leq c_2 \leq 8$ and there is an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0$, where $C$ is a smooth irreducible elliptic curve of degree $c_2$ or, if $c_2 = 8$, $C$ is the disjoint union of two smooth elliptic quartic curves.
Proof. A general section of $E$ vanishes along $C$ where $C$ is a smooth curve with $\omega_C = O_C$ and where $\mathcal{I}_C(4)$ is generated by global sections. Let $C = C_1 \cup \ldots \cup C_r$ be the decomposition into irreducible components: the union is disjoint, each $C_i$ is a smooth elliptic curve hence has degree at least three.

By Lemma 1.5 $d = \deg(C) \leq 8$. If $d \leq 4$ then $C$ is irreducible and is a complete intersection which is impossible since $E$ doesn’t split. If $d = 5$, $C$ is smooth irreducible.

Claim: If $8 \geq d \geq 6$, $C$ cannot contain a plane cubic curve.

Assume $C = P \cup X$ where $P$ is a plane cubic and where $X$ is a smooth elliptic curve of degree $d - 3$. If $d = 6$, $X$ is also a plane cubic and every quartic containing $C$ contains the line $\langle P \rangle \cap \langle X \rangle$. If $\deg(X) \geq 4$ then every quartic, $F$, containing $C$ contains the plane $\langle P \rangle$. Indeed $F|H$ vanishes on $P$ and on the $\deg(X) \geq 4$ points of $X \cap \langle P \rangle$, but these points are not on a line so $F|H = 0$. In both cases we get a contradiction with the fact that $\mathcal{I}_C(4)$ is generated by global sections. The claim is proved.

It follows that, if $8 \geq d \geq 6$, then $C$ is irreducible except if $C = X \cup Y$ is the disjoint union of two elliptic quartic curves.

Now let’s show that all possibilities of Proposition 1.6 do actually occur. For this we have to show the existence of a smooth irreducible elliptic curve of degree $d$, $5 \leq d \leq 8$ with $\mathcal{I}_C(4)$ generated by global sections (and also that the disjoint union of two elliptic quartic curves is cut off by quartics).

Lemma 1.7. There exist rank two vector bundles with $c_1 = 4, c_2 = 5$ which are globally generated. More precisely any such bundle is of the form $\mathcal{N}(2)$, where $\mathcal{N}$ is a null-correlation bundle (a stable bundle with $c_1 = 0, c_2 = 1$).

Proof. The existence is clear (if $\mathcal{N}$ is a null-correlation bundle then it is well known that $\mathcal{N}(k)$ is globally generated if $k \geq 1$). Conversely if $E$ has $c_1 = 4, c_2 = 5$ and is globally generated, then $E$ has a section vanishing along a smooth, irreducible quintic elliptic curve (cf 1.6). Since $h^0(\mathcal{I}_C(2)) = 0$, $E$ is stable, hence $E = \mathcal{N}(2)$. \qed

Lemma 1.8. There exist smooth, irreducible elliptic curves, $C$, of degree 6 with $\mathcal{I}_C(4)$ generated by global sections.

Proof. Let $X$ be the union of three skew lines. The curve $X$ lies on a smooth quadric surface, $Q$, and has $\mathcal{I}_X(3)$ globally generated (indeed the exact sequence
0 \to I_Q \to I_X \to I_{X,Q} \to 0 \text{ twisted by } O(3) \text{ reads like: } 0 \to O(1) \to I_C(3) \to O_Q(3,0) \to 0.

The complete intersection, $U$, of two general cubics containing $X$ links $X$ to a smooth curve, $C$, of degree 6 and arithmetic genus 1. Since, by liaison, $h^1(I_C) = h^1(I_X(-2)) = 0$, $C$ is irreducible. The exact sequence of liaison:

$0 \to I_U(4) \to I_C(4) \to \omega_X(2) \to 0$

shows that $I_C(4)$ is globally generated.

\[\square\]

In order to prove the existence of smooth, irreducible elliptic curves, $C$, of degree $d = 7, 8$, with $I_C(4)$ globally generated, we have to recall some results due to Mori ([10]).

According to [10] Remark 4, Prop. 6, there exists a smooth quartic surface $S \subset \mathbb{P}^3$ such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}X$ where $X$ is a smooth elliptic curve of degree $d$ $(7 \leq d \leq 8)$. The intersection pairing is given by:

$H^2 = 4, X^2 = 0, H.X = d.$

Such a surface doesn’t contain any smooth rational curve ([10] p.130). In particular:

\(\star\) every integral curve, $Z$, on $S$ has degree $\geq 4$ with equality if and only if $Z$ is a planar quartic curve or an elliptic quartic curve.

Lemma 1.9. With notations as above, $h^0(I_X(3)) = 0$.

Proof. A curve $Z \in |3H - X|$ has invariants $(d_Z, g_Z) = (5,-2)$ (if $d = 7$) or $(4,-5)$ (if $d = 8$), so $Z$ is not integral. It follows that $Z$ must contain an integral curve of degree $< 4$, but this is impossible.

\[\square\]

Lemma 1.10. With notations as above $|4H - X|$ is base point free, hence there exist smooth, irreducible elliptic curves, $X$, of degree $d$, $7 \leq d \leq 8$, such that $I_X(4)$ is globally generated.

Proof. Let’s first prove the following: Claim: Every curve in $|4H - X|$ is integral.

If $Y \in |4H - X|$ is not integral then $Y = Y_1 + Y_2$ where $Y_1$ is integral with $\deg(Y_1) = 4$ (observe that $\deg(Y) = 9$ or 8).

If $Y_1$ is planar then $Y_1 \sim H$, so $4H - X \sim H + Y_2$ and it follows that $3H \sim X + Y_2$, in contradiction with $h^0(I_X(3)) = 0$ (cf [10]).

So we may assume that $Y_1$ is a quartic elliptic curve, i.e. (i) $Y_1^2 = 0$ and (ii) $Y_1.H = 4$. Setting $Y_1 = aH + bX$, we get from (i): $2a(2a + bd) = 0$. Hence

(a) $a = 0$, or (b) $2a + bd = 0$.

(\alpha) In this case $Y_1 = bX$, hence (for degree reasons and since $S$ doesn’t contain curves of degree $< 4$), $Y_2 = \emptyset$ and $Y = X$, which is integral.

(\beta) Since $Y_1.H = 4$, we get $2a + (2a + bd) = 2a = 4$, hence $a = 2$ and $bd = -4$ which is impossible ($d = 7$ or 8 and $b \in \mathbb{Z}$).
This concludes the proof of the claim.

Since \((4H - X)^2 \geq 0\), the claim implies that \(4H - X\) is numerically effective. Now we conclude by a result of Saint-Donat (cf. [10], Theorem 5) that \(|4H - X|\) is base point free, i.e. \(\mathcal{I}_{X,S}(4)\) is globally generated. By the exact sequence: \(0 \to \mathcal{O} \to \mathcal{I}_X(4) \to \mathcal{I}_{X,S}(4) \to 0\) we get that \(\mathcal{I}_X(4)\) is globally generated. □

Remark 1.11. If \(d = 8\), a general element \(Y \in |4H - X|\) is a smooth elliptic curve of degree 8. By the way \(Y \neq X\) (see [1]). The exact sequence of liaison: \(0 \to \mathcal{I}_U(4) \to \mathcal{I}_X(4) \to \omega_Y \to 0\) shows that \(h^0(\mathcal{I}_X(4)) = 3\) (i.e. \(X\) is of maximal rank). In case \(d = 8\) Lemma 1.10 is stated in [2], however the proof there is incomplete, indeed in order to apply the enumerative formula of [8] one has to know that \(X\) is a connected component of \(\bigcap_{i=1}^3 F_i\); this amounts to say that the base locus of \(|4H - X|\) on \(F_1\) has dimension \(\leq 0\).

To conclude we have:

Lemma 1.12. Let \(X\) be the disjoint union of two smooth, irreducible quartic elliptic curves, then \(\mathcal{I}_X(4)\) is generated by global sections.

Proof. Let \(X = C_1 \sqcup C_2\). We have: \(0 \to \mathcal{O}(4) \to \mathcal{I}_X(4) \to \omega_Y \to 0\) shows that \(h^0(\mathcal{I}_X(4)) = 3\) (i.e. \(X\) is of maximal rank). In case \(d = 8\) Lemma 1.10 is stated in [2], however the proof there is incomplete, indeed in order to apply the enumerative formula of [8] one has to know that \(X\) is a connected component of \(\bigcap_{i=1}^3 F_i\); this amounts to say that the base locus of \(|4H - X|\) on \(F_1\) has dimension \(\leq 0\).

Summarizing:

Proposition 1.13. There exists an indecomposable rank two vector bundle, \(E\), on \(\mathbb{P}^3\), generated by global sections and with \(c_1(E) = 4\) if and only if \(5 \leq c_2(E) \leq 8\) and in these cases there is an exact sequence:

\[0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0\]

where \(C\) is a smooth irreducible elliptic curve of degree \(c_2(E)\) or, if \(c_2(E) = 8\), the disjoint union of two smooth elliptic quartic curves.

1.4. Globally generated rank two vector bundles with \(c_1 = 5\).

We start by listing the possible cases:

Proposition 1.14. If \(E\) is an indecomposable, globally generated, rank two vector bundle on \(\mathbb{P}^3\) with \(c_1(E) = 5\), then \(c_2(E) \in \{8, 10, 12\}\) and there is an exact
sequence:
\[ 0 \to \mathcal{O} \to E \to I_C(5) \to 0 \]
where \( C \) is a smooth, irreducible curve of degree \( d = c_2(E) \), with \( \omega_C \simeq \mathcal{O}_C(1) \).

In any case \( E \) is stable.

**Proof.** A general section of \( E \) vanishes along a smooth curve, \( C \), of degree \( d = c_2(E) \) with \( \omega_C \simeq \mathcal{O}_C(1) \). Hence every irreducible component, \( Y \), of \( C \) is a smooth, irreducible curve with \( \omega_Y \simeq \mathcal{O}_Y(1) \). In particular \( \deg(Y) = 2g(Y) - 2 \) is even and \( \deg(Y) \geq 4 \).

1. If \( d = 4 \), then \( C \) is a planar curve and \( E \) splits.
2. If \( d = 6 \), \( C \) is necessarily irreducible (of genus 4). It is well known that any such curve is a complete intersection \( (2,3) \), hence \( E \) splits.
3. If \( d = 8 \) and \( C \) is not irreducible, then \( C = P_1 \sqcup P_2 \), the disjoint union of two planar quartic curves. If \( L = \langle P_1 \rangle \cap \langle P_2 \rangle \), then every quintic containing \( C \) contains \( L \) in contradiction with the fact that \( I_C(5) \) is generated by global sections. Hence \( C \) is irreducible.
4. If \( d = 10 \) and \( C \) is not irreducible, then \( C = P \sqcup X \), where \( P \) is a planar curve of degree 4 and \( X \) is a degree 6 curve \( (X \) is a complete intersection \( (2,3) \)). Every quintic containing \( C \) vanishes on \( P \) and on the 8 points of \( X \cap \langle P \rangle \), since these 8 points are not on a line, the quintic vanishes on the plane \( \langle P \rangle \). This contradicts the fact that \( I_C(5) \) is globally generated.
5. If \( d = 12 \) and \( C \) is not irreducible we have three possibilities:
   a. \( C = P_1 \sqcup P_2 \sqcup P_3 \), \( P_i \) planar quartic curves
   b. \( C = X_1 \sqcup X_2 \), \( X_i \) complete intersection curves of types \( (2,3) \)
   c. \( C = Y \sqcup P \), \( Y \) a canonical curve of degree 8, \( P \) a planar curve of degree 4.

a. This case is impossible (consider the line \( \langle P_1 \rangle \cap \langle P_2 \rangle \)).

b. We have \( X_i = Q_i \cap F_i \). Let \( Z \) be the quartic curve \( Q_1 \cap Q_2 \). Then \( X_i \cap Z = F_i \cap Z \), i.e. \( X_i \) meets \( Z \) in 12 points. It follows that every quintic containing \( C \) meets \( Z \) in 24 points, hence such a quintic contains \( Z \). Again this contradicts the fact that \( I_C(5) \) is globally generated.

c. This case too is impossible: every quintic containing \( C \) vanishes on \( P \) and on the points \( \langle P \rangle \cap Y \), hence on \( \langle P \rangle \).

We conclude that if \( d = 12 \), \( C \) is irreducible.
The normalized bundle is \( E(-3) \), since in any case \( h^0(\mathcal{I}_C(2)) = 0 \) (every smooth irreducible subcanonical curve on a quadric surface is a complete intersection), \( E \) is stable. □

Now we turn to the existence part.

**Lemma 1.15.** There exist indecomposable rank two vector bundles on \( \mathbb{P}^3 \) with Chern classes \( c_1 = 5 \) and \( c_2 \in \{8, 10\} \) which are globally generated.

**Proof.** Let \( R = \bigsqcup_{i=1}^s L_i \) be the union of \( s \) disjoint lines, \( 2 \leq s \leq 3 \). We may perform a liaison \( (s, 3) \) and link \( R \) to \( K = \bigsqcup_{i=1}^s K_i \), the union of \( s \) disjoint conics. The exact sequence of liaison: 

\[
0 \rightarrow \mathcal{I}_U(4) \rightarrow \mathcal{I}_K(4) \rightarrow \omega_R(5 - s) \rightarrow 0
\]

shows that \( \mathcal{I}_K(4) \) is globally generated (n.b. \( 5 - s \geq 2 \)). 

Since \( \omega_K(1) \simeq \mathcal{O}_K \) we have an exact sequence: 

\[
0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}(2) \rightarrow \mathcal{I}_K(3) \rightarrow 0
\]

where \( \mathcal{E} \) is a rank two vector bundle with Chern classes \( c_1 = -1, c_2 = 2s - 2 \). Twisting by \( \mathcal{O}(1) \) we get: 

\[
0 \rightarrow \mathcal{O}(1) \rightarrow \mathcal{E}(3) \rightarrow \mathcal{I}_K(4) \rightarrow 0
\] (\( * \)). The Chern classes of \( \mathcal{E}(3) \) are \( c_1 = 5, c_2 = 2s + 4 \) (i.e. \( c_2 = 8, 10 \)). Since \( \mathcal{I}_K(4) \) is globally generated, it follows from (\( * \)) that \( \mathcal{E}(3) \) too, is generated by global sections. □

**Remark 1.16.**

1. If \( \mathcal{E} \) is as in the proof of Lemma 1.15 a general section of \( \mathcal{E}(3) \) vanishes along a smooth, irreducible (because \( h^1(\mathcal{E}(-2)) = 0 \)) canonical curve, \( C \), of genus \( 1 + c_2/2 \) (\( g = 5, 6 \)) such that \( \mathcal{I}_C(5) \) is globally generated. By construction these curves are not of maximal rank (\( h^0(\mathcal{I}_C(3)) = 1 \) if \( g = 5 \), \( h^0(\mathcal{I}_C(4)) = 2 \) if \( g = 6 \)). As explained in [9] §4 this is a general fact: no canonical curve of genus \( g, 5 \leq g \leq 6 \) in \( \mathbb{P}^3 \) is of maximal rank. We don’t know if this is still true for \( g = 7 \).


3. The proof of 1.15 breaks down with four conics: \( \mathcal{I}_K(4) \) is no longer globally generated, every quartic containing \( K \) vanishes along the lines \( L_i \) \( (5 - s = 1) \). Observe also that four disjoint lines always have a quadrisecant and hence are an exception to the normal generation conjecture (the homogeneous ideal is not generated in degree three as it should be).

**Remark 1.17.** The case \((c_1, c_2) = (5, 12)\) remains open. It can be shown that if \( E \) exists, a general section of \( E \) is linked, by a complete intersections of two
quintics, to a smooth, irreducible curve, \(X\), of degree 13, genus 10 having \(\omega_X(-1)\) as a base point free \(g_1^5\). One can prove the existence of curves \(X \subset \mathbb{P}^4\), smooth, irreducible, of degree 13, genus 10, with \(\omega_X(-1)\) a base point free pencil and lying on one quintic surface. But we are unable to show the existence of such a curve with \(h^0(I_X(5)) \geq 3\) (or even with \(h^0(I_X(5)) \geq 2\)). We believe that such bundles do not exist.

2. Globally generated rank two vector bundles on \(\mathbb{P}^n\), \(n \geq 4\).

For \(n \geq 4\) and \(c_1 \leq 5\) there is no surprise:

**Proposition 2.1.** Let \(E\) be a globally generated rank two vector bundle on \(\mathbb{P}^n\), \(n \geq 4\). If \(c_1(E) \leq 5\), then \(E\) splits.

**Proof.** It is enough to treat the case \(n = 4\). A general section of \(E\) vanishes along a smooth (irreducible) subcanonical surface, \(S: 0 \rightarrow \mathcal{O} \rightarrow E \rightarrow I_S(c_1) \rightarrow 0\). By [5], if \(c_1 \leq 4\), then \(S\) is a complete intersection and \(E\) splits. Assume now \(c_1 = 5\). Consider the restriction of \(E\) to a general hyperplane \(H\). If \(E\) doesn’t split, by [1.14] we get that the normalized Chern classes of \(E\) are: \(c_1 = -1, c_2 \in \{2, 4, 6\}\). By Schwarzenberger condition: \(c_2(c_2 + 2) \equiv 0 \pmod{12}\). The only possibilities are \(c_2 = 4\) or \(c_2 = 6\). If \(c_2 = 4\), since \(E\) is stable (because \(E|H\) is, see [1.14]), we have (3) that \(E\) is a Horrocks-Mumford bundle. But the Horrocks-Mumford bundle (with \(c_1 = 5\)) is not globally generated.

The case \(c_2 = 6\) is impossible: such a bundle would yield a smooth surface \(S \subset \mathbb{P}^4\), of degree 12 with \(\omega_S \simeq \mathcal{O}_S\), but the only smooth surface with \(\omega_S \simeq \mathcal{O}_S\) in \(\mathbb{P}^4\) is the abelian surface of degree 10 of Horrocks-Mumford. \(\square\)

**Remark 2.2.** For \(n > 4\) the results in [6] give stronger and stronger (as \(n\) increases) conditions for the existence of indecomposable rank two vector bundles generated by global sections.

Putting everything together, the proof of Theorem 0.1 is complete.

**References**


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