RANK TWO GLOBALLY GENERATED VECTOR BUNDLES WITH $c_1 \leq 5$.

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Abstract. We classify globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 3$, with $c_1 \leq 5$. The classification is complete but for one case ($n = 3$, $c_1 = 5$, $c_2 = 12$).

Introduction.

Vector bundles generated by global sections are basic objects in projective algebraic geometry. Globally generated line bundles correspond to morphisms to a projective space, more generally higher rank bundles correspond to morphism to (higher) Grassmann varieties. For this last point of view (that won't be touched in this paper) see[11] [12], [13]. Also globally generated vector bundles appear in a variety of problems ([7] just to make a single, recent example).

In this paper we classify globally generated rank two vector bundles on $\mathbb{P}^n$ (projective space over k, $\overline{k} = k$, $ch(k) = 0$), $n \geq 3$, with $c_1 \leq 5$. The result is:

Theorem 0.1. *Let E be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, generated by global sections with Chern classes c_1, c_2, $c_1 \leq 5$.*

1. *If $n \geq 4$, then E is the direct sum of two line bundles*
2. *If $n = 3$ and E is indecomposable, then*

$$(c_1, c_2) \in S = \{((2,2), (4,5), (4,6), (4,7), (4,8), (5,8), (5,10), (5,12)\}.$$

If E exists there is an exact sequence:

$$0 \to \mathcal{O} \to E \to \mathcal{I}_C(c_1) \to 0 \ (*)$$

where $C \subset \mathbb{P}^3$ is a smooth curve of degree c_2 with $\omega_C(4 - c_1) \simeq \mathcal{O}_C$. The curve C is irreducible, *except maybe if $(c_1, c_2) = (4, 8)$: in this case C can be either irreducible or the disjoint union of two smooth conics.*

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(3) *For every $(c_1, c_2) \in S$, $(c_1, c_2) \neq (5, 12)$, there exists a rank two vector bundle on $\mathbb{P}^3$ with Chern classes (c_1, c_2) which is globally generated (and with an exact sequence as in (2)).*

The classification is complete, but for one case: we are unable to say if there exist or not globally generated rank two vector bundles with Chern classes $c_1 = 5, c_2 = 12$ on $\mathbb{P}^3$.

1. Rank two vector bundles on $\mathbb{P}^3$.

1.1. **General facts.**

For completeness let's recall the following well known results:

Lemma 1.1. *Let E be a rank r vector bundle on $\mathbb{P}^n$, $n \geq 3$. Assume E is generated by global sections.*

(1) *If $c_1(E) = 0$, then $E \simeq r.\mathcal{O}$*
(2) *If $c_1(E) = 1$, then $E \simeq \mathcal{O}(1) \oplus (r-1).\mathcal{O}$ or $E \simeq T(-1) \oplus (r-n).\mathcal{O}$.*

Proof. If $L \subset \mathbb{P}^n$ is a line then $E|L \simeq \bigoplus_{i=1}^r \mathcal{O}_L(a_i)$ by a well known theorem and $a_i \geq 0, \forall i$ since E is globally generated. It turns out that in both cases: $E|L \simeq \mathcal{O}_L(c_1) \oplus (r-1).\mathcal{O}_L$ for every line L, i.e. E is uniform. Then (1) follows from a result of Van de Ven ([14]), while (2) follows from IV. Prop. 2.2 of [4]. □

Lemma 1.2. *Let E be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$. If E has a nowhere vanishing section then E splits. If E is generated by global sections and doesn't split then $h^0(E) \geq 3$ and a general section of E vanishes along a smooth curve, C, of degree $c_2(E)$ such that $\omega_C(4 - c_1) \simeq \mathcal{O}_C$. Moreover $\mathcal{I}_C(c_1)$ is generated by global sections.*

Lemma 1.3. *Let E be a non split rank two vector bundle on $\mathbb{P}^3$ with $c_1 = 2$. If E is generated by global sections then E is a null-correlation bundle.*

Proof. We have an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(2) \to 0$, where C is a smooth curve with $\omega_C(2) \simeq \mathcal{O}_C$. It follows that C is a disjoint union of lines. Since $h^0(\mathcal{I}_C(2)) \geq 2$, $d(C) \leq 2$. Finally $d(C) = 2$ because E doesn't split. □

This settles the classification of rank two globally generated vector bundles with $c_1(E) \leq 2$ on $\mathbb{P}^3$.

1.2. Globally generated rank two vector bundles with $c_1 = 3$.

The following result has been proved in [13] (with a different and longer proof).

Proposition 1.4. *Let E be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 3$ then E splits.*

Proof. Assume a general section vanishes in codimension two, then it vanishes along a smooth curve C such that $\omega_C \simeq \mathcal{O}_C(-1)$. Moreover $\mathcal{I}_C(3)$ is generated by global sections. We have $C = \cup_{i=1}^r C_i$ (disjoint union) where each C_i is smooth irreducible with $\omega_{C_i} \simeq \mathcal{O}_{C_i}(-1)$. It follows that each C_i is a smooth conic. If $r \geq 2$ let $L = \langle C_1\rangle \cap \langle C_2\rangle$ ($\langle C_i\rangle$ is the plane spanned by C_i). Every cubic containing C contains L (because it contains the four points $C_1 \cap L$, $C_2 \cap L$). This contradicts the fact that $\mathcal{I}_C(3)$ is globally generated. Hence $r = 1$ and $E = \mathcal{O}(1) \oplus \mathcal{O}(2)$. $\square$

1.3. Globally generated rank two vector bundles with $c_1 = 4$.

Let's start with a general result:

Lemma 1.5. *Let E be a non split rank two vector bundle on $\mathbb{P}^3$ with Chern classes c_1, c_2. If E is globally generated and if $c_1 \geq 4$ then:*

$$c_2 \leq \frac{2c_1^3 - 4c_1^2 + 2}{3c_1 - 4}.$$

Proof. By our assumptions a general section of E vanishes along a smooth curve, C, such that $\mathcal{I}_C(c_1)$ is generated by global sections. Let U be the complete intersections of two general surfaces containing C. Then U links C to a smooth curve, Y. We have $Y \neq \varnothing$ since E doesn't split. The exact sequence of liaison: $0 \to \mathcal{I}_U(c_1) \to \mathcal{I}_C(c_1) \to \omega_Y(4 - c_1) \to 0$ shows that $\omega_Y(4 - c_1)$ is generated by global sections. Hence $\deg(\omega_Y(4 - c_1)) \geq 0$. We have $\deg(\omega_Y(4 - c_1)) = 2g' - 2 + d'(4 - c_1)$ ($g' = p_a(Y)$, $d' = \deg(Y)$). So $g' \geq \frac{d'(c_1-4)+2}{2} \geq 0$ (because $c_1 \geq 4$). On the other hand, always by liaison, we have: $g' - g = \frac{1}{2}(d' - d)(2c_1 - 4)$ ($g = p_a(C)$, $d = \deg(C)$). Since $d' = c_1^2 - d$ and $g = \frac{d(c_1-4)}{2} + 1$ (because $\omega_C(4 - c_1) \simeq \mathcal{O}_C$), we get: $g' = 1 + \frac{d(c_1-4)}{2} + \frac{1}{2}(c_1^2 - 2d)(2c_1 - 4) \geq 0$ and the result follows. $\square$

Now we have:

Proposition 1.6. *Let E be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 4$ and if E doesn't split, then $5 \leq c_2 \leq 8$ and there is an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0$, where C is a smooth irreducible elliptic curve of degree c_2 or, if $c_2 = 8$, C is the disjoint union of two smooth elliptic quartic curves.*

Proof. A general section of E vanishes along C where C is a smooth curve with $\omega_C = \mathcal{O}_C$ and where $\mathcal{I}_C(4)$ is generated by global sections. Let $C = C_1 \cup ... \cup C_r$ be the decomposition into irreducible components: the union is disjoint, each C_i is a smooth elliptic curve hence has degree at least three.
By Lemma 1.5 $d = \deg(C) \leq 8$. If $d \leq 4$ then C is irreducible and is a complete intersection which is impossible since E doesn't split. If $d = 5$, C is smooth irreducible.
Claim: If $8 \geq d \geq 6$, C cannot contain a plane cubic curve.
Assume $C = P \cup X$ where P is a plane cubic and where X is a smooth elliptic curve of degree $d-3$. If $d = 6$, X is also a plane cubic and every quartic containing C contains the line $\langle P \rangle \cap \langle X \rangle$. If $deg(X) \geq 4$ then every quartic, F, containing C contains the plane $\langle P \rangle$. Indeed $F|H$ vanishes on P and on the $deg(X) \geq 4$ points of $X \cap \langle P \rangle$, but these points are not on a line so $F|H = 0$. In both cases we get a contradiction with the fact that $\mathcal{I}_C(4)$ is generated by global sections. The claim is proved.
It follows that, if $8 \geq d \geq 6$, then C is irreducible except if $C = X \cup Y$ is the disjoint union of two elliptic quartic curves. $\square$

Now let's show that all possibilities of Proposition 1.6 do actually occur. For this we have to show the existence of a smooth irreducible elliptic curve of degree d, $5 \leq d \leq 8$ with $\mathcal{I}_C(4)$ generated by global sections (and also that the disjoint union of two elliptic quartc curves is cut off by quartics).

Lemma 1.7. *There exist rank two vector bundles with $c_1 = 4, c_2 = 5$ which are globally generated. More precisely any such bundle is of the form $\mathcal{N}(2)$, where $\mathcal{N}$ is a null-correlation bundle (a stable bundle with $c_1 = 0, c_2 = 1$).*

Proof. The existence is clear (if $\mathcal{N}$ is a null-correlation bundle then it is well known that $\mathcal{N}(k)$ is globally generated if $k \geq 1$). Conversely if E has $c_1 = 4, c_2 = 5$ and is globally generated, then E has a section vanishing along a smooth, irreducible quintic elliptic curve (cf 1.6). Since $h^0(\mathcal{I}_C(2)) = 0$, E is stable, hence $E = \mathcal{N}(2)$. $\square$

Lemma 1.8. *There exist smooth, irreducible elliptic curves, C, of degree 6 with $\mathcal{I}_C(4)$ generated by global sections.*

Proof. Let X be the union of three skew lines. The curve X lies on a smooth quadric surface, Q, and has $\mathcal{I}_X(3)$ globally generated (indeed the exact sequence

$0 \to \mathcal{I}_Q \to \mathcal{I}_X \to \mathcal{I}_{X,Q} \to 0$ twisted by $\mathcal{O}(3)$ reads like: $0 \to \mathcal{O}(1) \to \mathcal{I}_C(3) \to \mathcal{O}_Q(3,0) \to 0$). The complete intersection, U, of two general cubics containing X links X to a smooth curve, C, of degree 6 and arithmetic genus 1. Since, by liaison, $h^1(\mathcal{I}_C) = h^1(\mathcal{I}_X(-2)) = 0$, C is irreducible. The exact sequence of liaison: $0 \to \mathcal{I}_U(4) \to \mathcal{I}_C(4) \to \omega_X(2) \to 0$ shows that $\mathcal{I}_C(4)$ is globally generated. $\square$

In order to prove the existence of smooth, irreducible elliptic curves, C, of degree $d = 7, 8$, with $\mathcal{I}_C(4)$ globally generated, we have to recall some results due to Mori ([10]).

According to [10] Remark 4, Prop. 6, there exists a smooth quartic surface $S \subset \mathbb{P}^3$ such that $Pic(S) = \mathbb{Z}H \oplus \mathbb{Z}X$ where X is a smooth elliptic curve of degree d ($7 \leq d \leq 8$). The intersection pairing is given by: $H^2 = 4$, $X^2 = 0$, $H.X = d$. Such a surface doesn't contain any smooth rational curve ([10] p.130). In particular: $(*)$ every integral curve, Z, on S has degree ≥ 4 with equality if and only if Z is a planar quartic curve or an elliptic quartic curve.

Lemma 1.9. *With notations as above, $h^0(\mathcal{I}_X(3)) = 0$.*

Proof. A curve $Z \in |3H - X|$ has invariants $(d_Z, g_Z) = (5, -2)$ (if $d = 7$) or $(4, -5)$ (if $d = 8$), so Z is not integral. It follows that Z must contain an integral curve of degree < 4, but this is impossible. $\square$

Lemma 1.10. *With notations as above $|4H - X|$ is base point free, hence there exist smooth, irreducible elliptic curves, X, of degree d, $7 \leq d \leq 8$, such that $\mathcal{I}_X(4)$ is globally generated.*

Proof. Let's first prove the following: *Claim:* Every curve in $|4H - X|$ is integral.

If $Y \in |4H - X|$ is not integral then $Y = Y_1 + Y_2$ where Y_1 is integral with $\deg(Y_1) = 4$ (observe that $\deg(Y) = 9$ or 8).

If Y_1 is planar then $Y_1 \sim H$, so $4H - X \sim H + Y_2$ and it follows that $3H \sim X + Y_2$, in contradiction with $h^0(\mathcal{I}_X(3)) = 0$ (cf 1.9).

So we may assume that Y_1 is a quartic elliptic curve, i.e. (i) $Y_1^2 = 0$ and (ii) $Y_1.H = 4$. Setting $Y_1 = aH + bX$, we get from (i): $2a(2a + bd) = 0$. Hence (α) $a = 0$, or (β) $2a + bd = 0$.

(α) In this case $Y_1 = bX$, hence (for degree reasons and since S doesn't contain curves of degree < 4), $Y_2 = \varnothing$ and $Y = X$, which is integral.

(β) Since $Y_1.H = 4$, we get $2a + (2a + bd) = 2a = 4$, hence $a = 2$ and $bd = -4$ which is impossible ($d = 7$ or 8 and $b \in \mathbb{Z}$).

This concludes the proof of the claim.

Since $(4H - X)^2 \geq 0$, the claim implies that $4H - X$ is numerically effective. Now we conclude by a result of Saint-Donat (cf [10], Theorem 5) that $|4H - X|$ is base point free, i.e. $\mathcal{I}_{X,S}(4)$ is globally generated. By the exact sequence: $0 \to \mathcal{O} \to \mathcal{I}_X(4) \to \mathcal{I}_{X,S}(4) \to 0$ we get that $\mathcal{I}_X(4)$ is globally generated. $\square$

Remark 1.11. If $d = 8$, a general element $Y \in |4H - X|$ is a smooth elliptic curve of degree 8. By the way $Y \neq X$ (see [1]). The exact sequence of liaison: $0 \to \mathcal{I}_U(4) \to \mathcal{I}_X(4) \to \omega_Y \to 0$ shows that $h^0(\mathcal{I}_X(4)) = 3$ (i.e. X is of maximal rank). In case $d = 8$ Lemma 1.10 is stated in [2], however the proof there is incomplete, indeed in order to apply the enumerative formula of [8] one has to know that X is a *connected* component of $\bigcap_{i=1}^{3} F_i$; this amounts to say that the base locus of $|4H - X|$ on F_1 has dimension ≤ 0.

To conclude we have:

Lemma 1.12. *Let X be the disjoint union of two smooth, irreducible quartic elliptic cuvres, then $\mathcal{I}_X(4)$ is generated by global sections.*

Proof. Let $X = C_1 \sqcup C_2$. We have: $0 \to \mathcal{O}(-4) \to 2.\mathcal{O}(-2) \to \mathcal{I}_{C_1} \to 0$, twisting by $\mathcal{I}_{C_2}$, since $C_1 \cap C_2 = \varnothing$, we get:
$0 \to \mathcal{I}_{C_2}(-4) \to 2.\mathcal{I}_{C_2}(-2) \to \mathcal{I}_X \to 0$ and the result follows. $\square$

Summarizing:

Proposition 1.13. *There exists an indecomposable rank two vector bundle, E, on $\mathbb{P}^3$, generated by global sections and with $c_1(E) = 4$ if and only if $5 \leq c_2(E) \leq 8$ and in these cases there is an exact sequence:*

$$0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0$$

where C is a smooth irreducible elliptic curve of degree $c_2(E)$ or, if $c_2(E) = 8$, the disjoint union of two smooth elliptic quartic curves.

1.4. **Globally generated rank two vector bundles with $c_1 = 5$.**

We start by listing the possible cases:

Proposition 1.14. *If E is an indecomposable, globally generated, rank two vector bundle on $\mathbb{P}^3$ with $c_1(E) = 5$, then $c_2(E) \in \{8, 10, 12\}$ and there is an exact* *sequence:*

$$0 \to \mathcal{O} \to E \to \mathcal{I}_C(5) \to 0$$

where C is a smooth, irreducible curve of degree $d = c_2(E)$, with $\omega_C \simeq \mathcal{O}_C(1)$.

In any case E is stable.

Proof. A general section of E vanishes along a smooth curve, C, of degree $d = c_2(E)$ with $\omega_C \simeq \mathcal{O}_C(1)$. Hence every irreducible component, Y, of C is a smooth, irreducible curve with $\omega_Y \simeq \mathcal{O}_Y(1)$. In particular $\deg(Y) = 2g(Y) - 2$ is even and $\deg(Y) \geq 4$.

(1) If $d = 4$, then C is a planar curve and E splits.
(2) If $d = 6$, C is necessarily irreducible (of genus 4). It is well known that any such curve is a complete intersection $(2, 3)$, hence E splits.
(3) If $d = 8$ and C is not irreducible, then $C = P_1 \sqcup P_2$, the disjoint union of two planar quartic curves. If $L = \langle P_1 \rangle \cap \langle P_2 \rangle$, then every quintic containing C contains L in contradiction with the fact that $\mathcal{I}_C(5)$ is generated by global sections. Hence C is irreducible.
(4) If $d = 10$ and C is not irreducible, then $C = P \sqcup X$, where P is a planar curve of degree 4 and where X is a degree 6 curve (X is a complete intersection $(2, 3)$). Every quintic containing C vanishes on P and on the 8 points of $X \cap \langle P \rangle$, since these 8 points are not on a line, the quintic vanishes on the plane $\langle P \rangle$. This contradicts the fact that $\mathcal{I}_C(5)$ is globally generated.
(5) If $d = 12$ and C is not irreducible we have three possibilities:
   (a) $C = P_1 \sqcup P_2 \sqcup P_3$, P_i planar quartic curves
   (b) $C = X_1 \sqcup X_2$, X_i complete intersection curves of types $(2, 3)$
   (c) $C = Y \sqcup P$, Y a canonical curve of degree 8, P a planar curve of degree 4.

   (a) This case is impossible (consider the line $\langle P_1 \rangle \cap \langle P_2 \rangle$).
   (b) We have $X_i = Q_i \cap F_i$. Let Z be the quartic curve $Q_1 \cap Q_2$. Then $X_i \cap Z = F_i \cap Z$, i.e. X_i meets Z in 12 points. It follows that every quintic containing C meets Z in 24 points, hence such a quintic contains Z. Again this contradicts the fact that $\mathcal{I}_C(5)$ is globally generated.
   (c) This case too is impossible: every quintic containing C vanishes on P and on the points $\langle P \rangle \cap Y$, hence on $\langle P \rangle$.

   We conclude that if $d = 12$, C is irreducible.

The normalized bundle is $E(-3)$, since in any case $h^0(\mathcal{I}_C(2)) = 0$ (every smooth irreducible subcanonical curve on a quadric surface is a complete intersection), E is stable. $\square$

Now we turn to the existence part.

Lemma 1.15. *There exist indecomposable rank two vector bundles on $\mathbb{P}^3$ with Chern classes $c_1 = 5$ and $c_2 \in \{8, 10\}$ which are globally generated.*

Proof. Let $R = \sqcup_{i=1}^{s} L_i$ be the union of s disjoint lines, $2 \leq s \leq 3$. We may perform a liaison $(s, 3)$ and link R to $K = \sqcup_{i=1}^{s} K_i$, the union of s disjoint conics. The exact sequence of liaison: $0 \to \mathcal{I}_U(4) \to \mathcal{I}_K(4) \to \omega_R(5 - s) \to 0$ shows that $\mathcal{I}_K(4)$ is globally generated (n.b. $5 - s \geq 2$).
Since $\omega_K(1) \simeq \mathcal{O}_K$ we have an exact sequence: $0 \to \mathcal{O} \to \mathcal{E}(2) \to \mathcal{I}_K(3) \to 0$, where $\mathcal{E}$ is a rank two vector bundle with Chern classes $c_1 = -1, c_2 = 2s - 2$. Twisting by $\mathcal{O}(1)$ we get: $0 \to \mathcal{O}(1) \to \mathcal{E}(3) \to \mathcal{I}_K(4) \to 0$ $(*)$. The Chern classes of $\mathcal{E}(3)$ are $c_1 = 5$, $c_2 = 2s + 4$ (i.e. $c_2 = 8, 10$). Since $\mathcal{I}_K(4)$ is globally generated, it follows from $(*)$ that $\mathcal{E}(3)$ too, is generated by global sections. $\square$

Remark 1.16.

(1) If $\mathcal{E}$ is as in the proof of Lemma 1.15 a general section of $\mathcal{E}(3)$ vanishes along a smooth, irreducible (because $h^1(\mathcal{E}(-2)) = 0$) canonical curve, C, of genus $1 + c_2/2$ ($g = 5, 6$) such that $\mathcal{I}_C(5)$ is globally generated. By construction these curves are not of maximal rank ($h^0(\mathcal{I}_C(3)) = 1$ if $g = 5$, $h^0(\mathcal{I}_C(4)) = 2$ if $g = 6$). As explained in [9] §4 this is a general fact: no canonical curve of genus $g, 5 \leq g \leq 6$ in $\mathbb{P}^3$ is of maximal rank. We don't know if this is still true for $g = 7$.
(2) According to [9] the general canonical curve of genus 6 lies on a unique quartic surface.
(3) The proof of 1.15 breaks down with four conics: $\mathcal{I}_K(4)$ is no longer globally generated, every quartic containing K vanishes along the lines L_i ($5-s = 1$). Observe also that four disjoint lines always have a quadrisecant and hence are an exception to the *normal generation conjecture*(the omogeneous ideal is not generated in degree three as it should be).

Remark 1.17. The case $(c_1, c_2) = (5, 12)$ remains open. It can be shown that if E exists, a general section of E is linked, by a complete intersections of two quintics, to a smooth, irreducible curve, X, of degree 13, genus 10 having $\omega_X(-1)$ as a base point free g^1_5. One can prove the existence of curves $X \subset \mathbb{P}^3$, smooth, irreducible, of degree 13, genus 10, with $\omega_X(-1)$ a base point free pencil and lying on *one* quintic surface. But we are unable to show the existence of such a curve with $h^0(\mathcal{I}_X(5)) \geq 3$ (or even with $h^0(\mathcal{I}_X(5)) \geq 2$). We believe that such bundles do not exist.

2. Globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 4$.

For $n \geq 4$ and $c_1 \leq 5$ there is no surprise:

Proposition 2.1. *Let E be a globally generated rank two vector bundle on $\mathbb{P}^n$, $n \geq 4$. If $c_1(E) \leq 5$, then E splits.*

Proof. It is enough to treat the case $n = 4$. A general section of E vanishes along a smooth (irreducible) subcanonical surface, S: $0 \to \mathcal{O} \to E \to \mathcal{I}_S(c_1) \to 0$. By [5], if $c_1 \leq 4$, then S is a complete intersection and E splits. Assume now $c_1 = 5$. Consider the restriction of E to a general hyperplane H. If E doesn't split, by 1.14 we get that the normalized Chern classes of E are: $c_1 = -1$, $c_2 \in \{2, 4, 6\}$. By Schwarzenberger condition: $c_2(c_2 + 2) \equiv 0$ (mod 12). The only possibilities are $c_2 = 4$ or $c_2 = 6$. If $c_2 = 4$, since E is stable (because $E|H$ is, see 1.14), we have ([3]) that E is a Horrocks-Mumford bundle. But the Horrocks-Mumford bundle (with $c_1 = 5$) is not globally generated.

The case $c_2 = 6$ is impossible: such a bundle would yield a smooth surface $S \subset \mathbb{P}^4$, of degree 12 with $\omega_S \simeq \mathcal{O}_S$, but the only smooth surface with $\omega_S \simeq \mathcal{O}_S$ in $\mathbb{P}^4$ is the abelian surface of degree 10 of Horrocks-Mumford. □

Remark 2.2. For $n > 4$ the results in [6] give stronger and stronger (as n increases) conditions for the existence of indecomposable rank two vector bundles generated by global sections.

Putting everything together, the proof of Theorem 0.1 is complete.

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