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Secant Varieties of Segre Varieties

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I. Introduction

Let $\mathbb{X} \subset \mathbb{P}^n$ be a reduced, irreducible, and non-degenerate projective variety.

Definition: Let $r \leq n$, then:

1) a *secant* $\mathbb{P}^r$ *to* $\mathbb{X}$ is any linear subspace $\mathbb{P}^r$ of $\mathbb{P}^n$ for which there are $P_0, \ldots, P_r \in \mathbb{X}$ for which

$$\langle P_0, \ldots, P_r \rangle = \mathbb{P}^r;$$

2) $Sec_r(\mathbb{X}) := \overline{\bigcup \{ \text{ all secant } \mathbb{P}^r\text{'s to } \mathbb{X}\}}$
(also sometimes written $\mathbb{X}^{r+1}$)

Basic Question:

How big is $Sec_r(\mathbb{X})$? i.e. what is its dimension?

There is an *expected dimension* by counting parameters:

$$\min\{n, (r+1)\dim \mathbb{X} + r\}.$$

When $\mathbb{X}$ has a secant variety whose dimension is less than expected we say that the secant variety is *deficient*, and the difference between the expected and actual dimension is denoted the *deficiency* of that secant variety.

Today I want to consider the Basic Question for the case of the *Segre Varieties* with their usual embedding in projective space.

I will use these examples as the "playing field" on which to explain our approach to the general problem of dealing with secant varieties. As you will see, our approach contrasts with that (say) of Chiantini and Ciliberto in that we are able to deal only with very special families of varieties: rational and strongly algebraically defined! in fact, intimitely connected to three of the most basic functors of algebra – the formation of the polynomial algebra (the Veronese varieties), the tensor product (the Segre Varieties) and the Exterior Algebra (the Grassmann variety) as well as various combinations of these functors. One advantage of our approach, for these special varieties, is that we are often able to find deficient varieties and (perhaps more significantly) can often say with certainty that certain varieties are NOT deficient.

In the particular case of the Segre varieties you will be able to see the two main aspects of our approach: the combinatorial and the algebraic.

The combinatorial aspect will be present in our study of particular monomial ideals in polynomial rings.

The algebraic aspect will be evident when we translate the secant variety problem into one which requires that we deal with the Hilbert function of certain "fat point" schemes in projective space, or products of projective spaces. The algebra here is mainly in the study of the cohomology of these fat point schemes.

II. Notation and First Results

Let's begin by setting up the basic notation. If we set $\underline{n} = (n_1, \ldots, n_t)$, recall that the Segre Varieties are the varieties $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} = \mathbb{P}^{\underline{n}}$ embedded in $\mathbb{P}^{N_{\underline{n}}}$, $N_{\underline{n}} = \Pi(n_i + 1) - 1$. Let me recall how the embeddings are defined.

Let $V_1, \ldots, V_t$ be vector spaces over the field $k = \overline{k}$ where $\dim V_i = n_i + 1$. Think of $\mathbb{P}^{n_i} = \mathbb{P}(V_i)$. Then, the Segre embedding is based on the canonical multilinear map:

$$V_1 \times \cdots \times V_t \to V_1 \otimes \cdots \otimes V_t$$

i.e.

$$[v_1] \times \cdots \times [v_t] \longrightarrow [v_1 \otimes \cdots \otimes v_t].$$

and we are thinking of the enveloping $\mathbb{P}^{N_{\underline{n}}}$ as

$$\mathbb{P}(V_1 \otimes \cdots \otimes V_t).$$

Denote the embedded Segre Variety above by $\mathbb{X}_{\underline{n}}$.

With this point of view:

i) $\mathbb{X}_{\underline{n}}$ is the set of all classes of *decomposable* tensors, i.e. classes of tensors in $\mathbb{P}(V_1 \otimes \cdots \otimes V_t)$ of the form $[v_1 \otimes \cdots \otimes v_t]$.

ii) the secant variety, $Sec_r(\mathbb{X}_{\underline{n}})$, is the closure of the set of classes of those tensors which can be written as the sum of $\leq r+1$ decomposable tensors.

III. Two Factors

The case when $\underline{n} = (n_1, n_2)$.

After choosing bases for the underlying vector spaces we can identify

$$V_1 \otimes V_2 \leftrightarrow (n_1+1) \times (n_2+1) \ \text{ matrices,}$$

$$\text{decomposable vectors} \leftrightarrow \text{matrices of rank 1.}$$

With this point of view, the enveloping space for $\mathbb{X}_{\underline{n}}$ is the space of all these matrices (under projective equivalence) and the Segre variety consists of the classes of matrices of rank 1.

It is a standard fact of linear algebra that a matrix has rank $\leq r+1$ if and only if it is a sum of $\leq r+1$ matrices of rank 1.

Thus, if $\underline{n} = (n_1, n_2)$ the secant varieties of $\mathbb{X}_{\underline{n}}$ are the rank varieties of the generic $(n_1+1)\times(n_2+1)$ matrix.

Example: Consider the embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$, where $\mathbb{P}^8$ is the projective space of 3×3 matrices under projective equivalence.

The ideal of 2×2 minors of the generic matrix of size 3×3 defines $\mathbb{X}_{(2,2)}$ and the determinant of the generic matrix gives the equation of $Sec_1(\mathbb{X}_{(2,2)})$.

This shows that $\dim Sec_1(\mathbb{X}_{(2,2)}) = 7$. The expected dimension was:

$$\min\{8, 2(4)+1\} = 8,$$

i.e. we expected the secant variety to fill the enveloping space – but it does not. This is an example of a *deficient* Segre Variety.

In fact, if we assume that $n_1 \leq n_2$ then: for all r, $1 \leq r < n_1$ the secant varieties $Sec_r(\mathbb{X}_{\underline{n}})$ all have dimension less than the expected dimension. Moreover, the least integer r for which $Sec_r(\mathbb{X}_{\underline{n}})$ fills its enveloping space is $r = n_1$.

So, for two factors everything is pretty much known. One has a nice description of the varieties in question. One knows they are all arithmetically Cohen-Macaulay and one even has a description of the minimal free resolution of the defining ideal.

IV. More than Two Factors

For simplicity in notation, today I will mostly concentrate on the case of three factors (although almost everything I say is true for any number of factors ≥ 3). In fact, even for three factors there are many open problems; with some of the very interesting applications in this special case.

As mentioned earlier, we are interested in writing tensors as the sum of decomposable tensors. In particular we would like to know when the generic tensor is in $Sec_r(\mathbb{X}_{\underline{n}})$, i.e when this secant variety fills the enveloping space.

E.g. if A is some finite dimensional algebra over k the multiplication map in A is a bilinear map

$$A \times A \to A.$$

As such it gives rise to a particular tensor in

$$A^* \otimes A^* \otimes A.$$

If we knew how to express this tensor as a (short) sum of decomposable tensors, this would give us an algorithm for the multiplication in A.

For example, it has been shown that for the algebra of 2×2 matrices the tensor in question is the sum of 7 (and no fewer) decomposable tensors. This has resulted in a cheaper way to do matrix multiplication than the way we ordinarily do it. So, solutions to this sort of problem, even for the case of 3×3 matrices would actually make money!! In this case the tensor in quesion is in $\mathbb{P}^{8^*} \otimes \mathbb{P}^{8^*} \otimes \mathbb{P}^8 \subset \mathbb{P}^{728}$!

V. The Basic Geometric Result

Lemma: (Terracini) Let $\mathbb{X} \subset \mathbb{P}^n$ be as above. The dimension of $\mathbb{X}^{r+1}$ is the same as the dimension of the linear span of the tangent spaces to $r+1$ general points of $\mathbb{X}$.

(...because that linear space **IS** the tangent space to $Sec_r(\mathbb{X})$ at a general point of the secant $\mathbb{P}^r$ generated by those $r+1$ general points.)

From Terracini's Lemma, the obvious first step is to try and see if we can recognize the tangent space at a point on the Segre Variety.

In order to do that we have to introduce some (hideous) notation.

Choose bases for the three vector spaces as:

$$V_i = \langle x_{0,i}, x_{1,i}, \ldots, x_{n_i,i} \rangle.$$

Consider the three polynomial rings

$$S^i = k[x_{0,i}, x_{1,i}, \ldots, x_{n_i,i}],$$

and the $\mathbb{N}^3$ graded polynomial ring

$$A = k[x_{0,1}, \ldots, x_{n_1,1}; x_{0,2}, \ldots, x_{n_2,2}; x_{0,3}, \ldots, x_{n_3,3}],$$

where

$$\deg x_{i,1} = (1,0,0),$$

$$\deg x_{i,2} = (0,1,0)\ .$$

$$\deg x_{i,3} = (0,0,1)$$

Note that $\mathbf{V} = V_1 \otimes V_2 \otimes V_3$ is being identified with $A_{(1,1,1)}$ and we are embedding the product $\mathbb{P}^{\underline{n}}$, $\underline{n} = (n_1, n_2, n_3)$ in $\mathbb{P}(A_{(1,1,1)})$.

Consider the image of the point

$$P = [1:0:\ldots:0] \times [1:0:\ldots:0] \times [1:0:\ldots:0]$$

in $\mathbb{X}_{\underline{n}}$ and the tangent space to it. It is not difficult to show that the (*affine cone on the*) tangent space at that point (viewed in the polynomial ring A) is

$$[(S^1)]_1(x_{0,2}x_{0,3}) + [(S^2)]_1(x_{0,1}x_{0,3}) + [(S^3)]_1 x_{0,1}x_{0,2}$$

$$:= \mathbf{W}_{(1,1,1)}.$$

If we form the polynomial ring, B,

$$B = k[y_{0,1},\ldots,y_{n_1,1}; y_{0,2},\ldots,y_{n_2,2}; y_{0,3},\ldots,y_{n_3,3}],$$

where

$$\deg y_{i,1} = (1,0,0),$$

$$\deg y_{i,2} = (0,1,0)\ .$$

$$\deg y_{i,3} = (0,0,1)$$

then there is a natural duality between the two rings, where

$$y_{a,j} \circ x_{b,\ell} = (\partial/\partial x_{a,j})(x_{b,\ell}).$$

Under this duality we have a perfect pairing

$$B_{(1,1,1)} \times A_{(1,1,1)} \to k.$$

Using this pairing one obtains:

$$(\mathbf{W}_{(1,1,1)})^{\perp} =$$

$$[(y_{1,1},..,y_{n_1,1};y_{1,2},\dots,y_{n_2,2};y_{1,3},\dots,y_{n_3,3})^2]_{(1,1,1)}.$$

I.e. the degree one piece of the ideal I_P^2, considered in B.

Putting together the various pieces we have:

the dimension of the tangent space at a point P in $\mathbb{X}_{\underline{n}}$ is:

$$\dim_k(B_{(1,1,1)}/(I_P^2)_{(1,1,1)}) - 1 = (n_1 + n_2 + n_3 + 1) - 1$$

$$= n_1 + n_2 + n_3.$$

(which we already knew! it's the dimension of $\mathbb{X}_{\underline{n}}$).

We shall call a subscheme of $\mathbb{P}^{\underline{n}}$ defined by an ideal of the form I_P^2 a *2-fat point in* $\mathbb{P}^{\underline{n}}$.

Since $^\perp$ converts $+$'s into $\cap$'s we obtain:

let $P_i = p_{i1} \times p_{i2} \times p_{i3}$ $(i = 1, \ldots, s)$ be generic points in $\mathbb{P}^{\underline{n}}$. Let $I_{P_i} \subset B$ be the defining ideal of P_i and set

$$I = I_{P_1}^2 \cap \ldots \cap I_{P_s}^2 \subseteq B.$$

Let $\mathcal{P}_i$ be the images of P_i in $\mathbb{X}_{\underline{n}}$ and let W_i be the affine cone over the tangent space to $\mathcal{P}_i$.

Then

$$(W_1 + \cdots W_s)^\perp = (I)_{(1,1,1)}$$

and so (by Terracini)

$$\dim Sec_{s-1}(\mathbb{X}_{\underline{n}}) = \dim_k(B_{(1,1,1)}/I_{(1,1,1)}) - 1.$$

All of what I said above is true also in any dimension, so the question of determining the dimension of $\mathbb{X}^s_{\underline{n}}$ is completely dependent on finding the Hilbert function (in degree $(1,\ldots,1)$) of s generic 2-fat points in $\mathbb{P}^{\underline{n}}$.

There are two ways to approach this problem:

1) You think that $\dim \mathbb{X}^s_{\underline{n}}$ is the expected one! In that case, it is enough to find **ANY** set of s 2-fat points which impose the correct number of independent conditions to the forms of degree $(1,\ldots,1)$. I.e. you try to prove that they impose

$$\min\{\ s(n_1+\ldots+n_t+1), (n_1+1)(n_2+1)\cdots(n_t+1)\ \}$$

conditions to the forms of degree $(1,\ldots,1)$ in A.

OR

2) You think that the dimension of $\mathbb{X}^s_{\underline{n}}$ is *not* the expected one. In that case, you have to prove that

for **ANY** set of s 2-fat points there are more forms of degree $(1, \ldots, 1)$ vanishing on that scheme than there should be.

We can, in certain cases, do both of these things combinatorially. Today I'll only consider times when the combinatorial approach shows that the *expected dimension* of the secant varieties is the actual dimension. (For simplicity in the notation I'll go back to the case of three factors)!

Let $P \in \mathbb{P}^{\underline{n}}$, $\underline{n} = (n_1, n_2, n_3)$ and write $P = (p_1, p_2, p_3)$.

Def. P is a *coordinate point* of $\mathbb{P}^{\underline{n}}$ if each p_i is a coordinate point in $\mathbb{P}^{n_i}$.

There are $\Pi_{i=1}^{3}(n_i + 1)$ coordinate points in $\mathbb{P}^{\underline{n}}$ and we can think of them as corresponding to places on a 3-dimensional chessboard; where the position (i_1, i_2, i_3) on the chessboard corresponds to the product of the i_1 coordinate point in $\mathbb{P}^{n_1}$, the i_2 coordinate point in $\mathbb{P}^{n_2}$ and the i_3 coordinate point in $\mathbb{P}^{n_3}$.

It is easy to see that the ideal of B corresponding to the coordinate point $P \leftrightarrow (i_1, i_2, i_3)$ is $I_P =$

$$(y_{0,1}, \ldots, \widehat{y_{i_1,1}}, \ldots, y_{n_1,1}; y_{0,2}, \ldots, \widehat{y_{i_2,2}}, \ldots, y_{n_2,2};$$

$$y_{0,3}, \ldots, \widehat{y_{i_3,3}}, \ldots, y_{n_3,3}).$$

I_P is a monomial ideal and hence so is I_P^2. Let's try to describe this latter ideal, at least in degree $(1,1,1)$.

A monomial of degree $(1,1,1)$ looks like

$$M = y_{j_1,1}\ y_{j_2,2}\ y_{j_3,3}.$$

Such a monomial is in $I_P^2 \Leftrightarrow$ at least 2 of the entries in (j_1, j_2, j_3) are different from (i_1, i_2, i_3).

Put another way, M is not in $I_P^2 \Leftrightarrow$ at most one of (j_1, j_2, j_3) is different from (i_1, i_2, i_3).

It is easy to picture this situation on the chessboard!

Place a rook at the point (i_1, i_2, i_3) on the chessboard. The places that rook can attack correspond precisely to the places (j_1, j_2, j_3) which are different from (i_1, i_2, i_3) in at most **one** place. How many such places are there? exactly $(n_1 + n_2 + n_3) + 1$ (the "1" coming from the place where the rook is sitting already!).

Let's introduce some *chessboard notation*:

a set $R = \{P_1, \ldots, P_s\}$ of coordinate points in $\mathbb{P}^{\underline{n}}$ will be called a *rook set* and let $\langle R \rangle$ denote all the points on the chessboard which can be attacked by rooks placed at the points on the chessboard corresponding to the P_i. If we let $Z \subset \mathbb{P}^{\underline{n}}$ be the scheme of 2-fat points whose support is R then

$$H(Z, (1, 1, 1)) = \mid \langle R \rangle \mid .$$

We say that a rook set is *perfect* if every element in $\langle R \rangle$ is attacked by exactly one member of R.

Notice that there are no perfect rook sets on a two dimensional chessboard but there is an easy one to see on a $2 \times 2 \times 2$ chessboard!

Theorem: Let $\underline{n} = (n_1, n_2, n_3)$. If the $(n_1 + 1) \times (n_2 + 1) \times (n_3 + 1)$ chessboard supports a perfect rook set with s rooks then $\dim \mathbb{X}^s_{\underline{n}}$ is the expected dimension.

Corollary: If $s \leq n_1 + 1$ then $\mathbb{X}^s_{\underline{n}}$ has the expected dimension.

(place the rooks on the diagonal!).

So, (roughly speaking) the "*small*" secant varieties for Segre's are never deficient. In particular:

Corollary: The secant line variety for any Segre variety has the expected dimension.

Remarks:

1) It's clear that the discussion above is ok for any $\underline{n}$, not just for three factors.

2) If all the $n_i + 1 = q$, a fixed value, then problems about rook sets can be translated into problems in coding theory. In this case, a perfect rook set with s rooks is a 1-correcting code with s words and an alphabet of q letters. (explain!).

3) Perfect rook sets R for which $\langle R \rangle$ is the entire chessboard correspond to *perfect codes*. There has been a long and difficult search for perfect codes. They are known to exist only for the following parameters:

$$q \text{ a prime power }, t = \frac{q^k - 1}{q - 1}(k \geq 2), s = q^{t-k}.$$

(One proves the existence of such codes using the existence of finite fields with q elements. I.e. the theory of finite fields is useful for us in studying the algebraic geometry of high dimensional complex projective spaces!!!!)

But, that is very interesting for us, since we can use their existence to make assertions about secant varieties of Segre varieties.

Example:

1) Let k be any positive integer, $q = 2$, $t = 2^k - 1$, $s = 2^{t-k}$. For these numbers we find that for the Segre embedding

$$\mathbb{X}_t = \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{t=2^k-1 \text{ -times}} \to \mathbb{P}^{2^{2k-1}-1} = \mathbb{P}^{2^t-1}$$

we have $Sec_{s-1}(\mathbb{X}_t) = \mathbb{P}^{2^t-1}$ and these secant varieties fit "exactly" into their enveloping space.

It follows that **all** the secant varieties of $\mathbb{X}_t$ have the expected dimension.

It is worth noting here that the t's above are the only ones with the property that the generic tensor in the tensor product of t copies of a 2-dimensional vector space should be expressible in only a finite number of ways as a sum of s decomposable tensors. One wonders if it is true that for any of these values of t there is a *unique* such decomposition for a generic tensor. I.e. is it true that a generic member of the envelopping $\mathbb{P}^{2^t-1}$ lies on exactly one secant $\mathbb{P}^{s-1}$ to $\mathbb{X}_t$?

2) We can make families of similar examples for products of $\mathbb{P}^2,\ \mathbb{P}^3,\ \mathbb{P}^4,\ \mathbb{P}^7,\ \mathbb{P}^8,\ \ldots, \mathbb{P}^{q-1}$ where q is a prime power.

Given such a q, for any integer $k \geq 1$ we take $t = (q^k - 1)/(q-1)$ copies of $\mathbb{P}^{q-1}$, which gets embedded in $\mathbb{P}^{q^t-1}$. Then, for $s = q^{t-k}$ we get

$$Sec_{s-1}(\underbrace{\mathbb{P}^{q-1} \times \cdots \times \mathbb{P}^{q-1}}_{t \text{ -times}}) = \mathbb{P}^{q^t-1}$$

exactly!.

Question: It would be very nice if, in the family of all products of a **fixed** $\mathbb{P}^n$, there were only a finite number of deficient secant varieties. Is that true?

I'll come back to this question shortly for products of $\mathbb{P}^1$'s, the first case to consider.

The Simultaneous Homogenization Method

Apart from the cases where we could deal with fat point schemes by looking at monomial ideals, trying to understand the multigraded Hilbert functions of generic 2-fat point schemes in $\mathbb{P}^{\underline{n}}$ seems very difficult.

This fact forced us to search for other methods for trying to deal with 2-fat point schemes.

One method which we have found that does some cases very well is the one I want to briefly explain now. Roughly speaking we look at $\mathbb{P}^{\underline{n}}$ and then "dehomogenize" each factor $\mathbb{P}^{n_i}$ to $\mathbb{A}^{n_i}$. Since we have only a finite number of points, we consider them as lying in this affine piece of $\mathbb{P}^{\underline{n}}$, which is an affine space $\mathbb{A}^{n_1+\cdots+n_t}$. Now homogenize and keep track of things in this new $\mathbb{P}^{n_1+\cdots+n_t}$.

Recall: if $\underline{n} = (n_1, \ldots, n_t)$ then $\mathbb{X}_{\underline{n}}$ is the image of

$$\tau_{\underline{n}} : \mathbb{P}^{\underline{n}} \longrightarrow \mathbb{P}^{N_{\underline{n}}}$$

(the Segre embedding). Let $P_1, \ldots, P_s$ be s generic points in $\mathbb{P}^{\underline{n}}$ and let $\mathbb{Z}$ be the scheme defined by

$$I_{\mathbb{Z}} = (I_{P_1})^2 \cap \ldots \cap (I_{P_s})^2 \subset B$$

where

$$B = k[y_{0,1}, \ldots, y_{n_1,1}; y_{0,2}, \ldots, y_{n_2,2};$$

$$\ldots; y_{0,t}, \ldots, y_{n_t,t}].$$

Then

$$\dim \mathbb{X}^s_{\underline{n}} = \dim Sec_{s-1}(\mathbb{X}_{\underline{n}}) =$$

$$= \dim_k \left(\frac{B}{I_{\mathbb{Z}}}\right)_{(1,\ldots,1)} - 1$$

□

So, if $\mathbb{Z}$ is a scheme of s generic 2-fat points in $\mathbb{P}^{\underline{n}}$ there is no loss in assuming that $\mathbb{Z}$ is contained in the affine chart defined by $y_{0,1} \cdots y_{0,t} \neq 0$. Let $n = n_1 + \cdots + n_t$. We seek a scheme $W \subset \mathbb{P}^n$ such that

$$\dim(I_W)_t = \dim(I_{\mathbb{Z}})_{(1,\ldots,1)}.$$

Without going into the details (which are simple, but notationally messy) we obtain the following:

let

$$S = k[z_0, z_{1,1}, \ldots, z_{n_1,1}, \ldots, z_{1,t}, \ldots, z_{n_t,t}]$$

be the homogeneous coordinate ring of $\mathbb{P}^n$ and let

$$Q_0, Q_{1,1}, \ldots, Q_{n_1,1}, \ldots, Q_{1,t}, \ldots, Q_{n_t,t}$$

be the coordinate points of $\mathbb{P}^n$. Set

$$\Pi_i = \langle Q_{1,i}, \ldots, Q_{n_i,i} \rangle$$

and let W_i be the subscheme of $\mathbb{P}^n$ which is $(t-1)\Pi_i$ (i.e. if I_{Π_i} is the prime ideal of S corresponding to Π_i, then W_i is the subscheme of $\mathbb{P}^n$ defined by $I_{\Pi_i}^{t-1}$).

If we let Z' be a subscheme of s generic 2-fat points in $\mathbb{P}^n$, then:

Theorem: If $W = Z' + W_1 + \ldots + W_t \subset \mathbb{P}^n$,

$$\dim_k (I_W)_t = \dim (I_Z)_{(1,\ldots,1)}.$$

The key point is: we replaced a scheme in a multiprojective space by a scheme in standard projective space. Consequently, all the machinery in dealing with such schemes is now available to us.

Example: Let me illustrate the importance of this switch in a very particular case: the product of $\mathbb{P}^1$ t-times (which I'll write $(\mathbb{P}^1)^t$). We'll denote by $\mathbb{X}_t$ its embedding in $\mathbb{P}^{2^t-1}$.

The method says that we can replace a scheme $\mathbb{Z}$ of s generic 2-fat points in $(\mathbb{P}^1)^t$ with the following scheme in $\mathbb{P}^t$:

the union of
i) s generic 2-fat points in $\mathbb{P}^t$;
ii) t other schemes, $W_1, \ldots, W_t$; where if $Q_0, \ldots, Q_t$ are the coordinate points in $\mathbb{P}^t$, then

$$W_i = (t-1)Q_i.$$

E.g. for $t = 4$ we have $\mathbb{X}_4 = \mathbb{X}_{(1,1,1,1)} \subset \mathbb{P}^{15}$, then $\dim_k[\mathbb{X}_4^3 = Sec_2(\mathbb{X}_4)]$ is

$$(2^4-1) - \dim_k(\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap q_1^3 \cap q_2^3 \cap q_3^3 \cap q_4^3)_4$$

where the q_i are the prime ideals of coordinate points in $\mathbb{P}^4$ and the $\wp_i$ are the prime ideals of general points. There are, then, 7 general points of $\mathbb{P}^4$.

Since 7 points of $\mathbb{P}^4$ are always on a rational normal curve, the dimensions of all the pieces of such ideals are known and have been computed in a paper by Catalisano, Ellia, and Gimigliano. One expects this ideal to contain only one form of degree 4 but they show it has two. This has, as a consequence, that $\mathbb{X}_4$ has a deficient secant plane variety.

This appears to be the only deficient secant variety in the entire family of products of $\mathbb{P}^1$'s.

In fact, in a paper by Catalisano, Gimigliano and I we showed (using the ideas above):

Theorem:

Let $\mathbb{X}_t$ be the Segre embedding of $(\mathbb{P}^1)^t$ in $\mathbb{P}^N$, $N = 2^t - 1$.

Let $e_t = [2^t/(t+1)] \simeq \delta_t (\mathrm{mod}\ 2)$ and define

$$s_t = e_t - \delta_t.$$

If $s \neq s_t + 1$ then

$$\dim \mathbb{X}_t^s = \min\{s(t+1); N\}$$

i.e. the secant variety has the expected dimension in all these cases.

Indications of the Inductive Proof:

Special Case of $(\mathbb{P}^1)^5 \subset \mathbb{P}^{31}$ as $\mathbb{X}_5$.

Need to show: for

$$s_5 = 4 \qquad \text{and} \qquad s_5 + 2 = 6$$

$$\dim Sec_3(\mathbb{X}_5) = 23 \quad \text{and} \quad \dim Sec_5(\mathbb{X}_5) = 31$$

the expected dimensions.

Note that the Theorem makes no statement about the dimension of $Sec_4(\mathbb{X}_5)$, which has expected dimension 29. (We've verified, using CoCoA, that this dimension is also correct.)

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Some simple algebraic facts we will need are:

if $\wp = (x_1, \ldots, x_n) \subset R = k[x_0, \ldots, x_n]$ then

$i)$ $\wp^\ell$ is a $\wp$-primary ideal (Macaulay);
$ii)$ $(\wp^\ell : x_0) = \wp^\ell$;
$iii)$ $(\wp^\ell : x_1) = \wp^{\ell-1}$.

Also: let $I = I_{\mathbb{Z}}$ be the (saturated) ideal of R defining the subscheme $\mathbb{Z} \subset \mathbb{P}^n$. Let F be a homogeneous form of degree d. Then:

i) the scheme defined by $(I : F)$ (automatically saturated) is called the *residual* of $\mathbb{Z}$ with respect to $\mathcal{D} = V(F)$ and denoted

$$Res_{\mathcal{D}}(\mathbb{Z}) = Z'.$$

ii) The subscheme of $\mathcal{D}$ defined by the ideal $(I, F)/(F)$ in $R/(F)$ (not necessarily saturated) is called the *trace of* $\mathbb{Z}$ *on* $\mathcal{D}$ and denoted

$$Tr_{\mathcal{D}}(\mathbb{Z}) = \mathbb{Z}''.$$

For $t \geq d$ we have the *Castelnuovo Inequality:*

$$\dim_k(I_{\mathbb{Z},\mathbb{P}^n})_t \leq \dim_k(I_{\mathbb{Z}',\mathbb{P}^n})_{t-d} + \dim_k(I_{\mathbb{Z}'',\mathcal{D}})_t$$

which is an immediate consequence of the exact sequence

$$0 \to (I : F)(-d) \xrightarrow{\times F} I \to \frac{I}{F(I : F)} \to 0$$

and the observations:

$a)$ $F(I : F) = (F) \cap I$ and

$b)$ $\frac{I}{F(I:F)} = \frac{(I,F)}{(F)}$.

Now,taking cohomology, and noting that $\frac{(I,F)}{(F)}$ need not be saturated, the inequality follows.

The major algebraic fact we will use is the incredible "Differential Horace Lemma" of J. Alexander and A. Hirschowitz.

Lemma: Let $H \subset \mathbb{P}^n$ be a hypersurface, $P_1, \ldots, P_r$ generic points of $\mathbb{P}^n$.

Consider a zero dimensional scheme $\mathbb{Z} \subset \mathbb{P}^n$,

$$\mathbb{Z} = \tilde{\mathbb{Z}} + 2P_1 + \cdots + 2P_r.$$

Form $Res_H(\tilde{\mathbb{Z}}) = \tilde{Z}'$ and $Tr_H(\tilde{Z}) = \tilde{Z}''$.

If $P_1', \cdots, P_r'$ are generic points in H, set

$$D_{2,H} = 2P_i' \cap H(\text{ a degree } n \text{ subscheme of } \mathbb{P}^n).$$

Consider the two schemes: (called respectively *Degue* and *Dime* by [A-H]:)

$$\mathbb{Z}' = \tilde{Z}' + D_{2,H}(P_1') + \cdots + D_{2,H}(P_r') \subset \mathbb{P}^n$$

$$\mathbb{Z}'' = \tilde{Z}'' + P_1' + \cdots + P_r' \subset \mathbb{P}^{n-1} = H.$$

If $\dim_k(I_{\mathbb{Z}'})_{t-1} = 0$ and $\dim_k(I_{Z''})_t = 0$ then

$$\dim_k(I_Z)_t = 0.$$

Back to our special case: to use the multihomogenous dehomogenization method to find the dimension of $Sec_3(\mathbb{X}_5)$ we must consider the scheme

$$W = 4Q_1 + \ldots + 4Q_5 + 2P_1 + \cdots + 2P_4 \in \mathbb{P}^5$$

(where $Q_1, \ldots, Q_5$ are the coordinate points of $\mathbb{P}^5$ for which $x_0 = 0$ and $P_1, \ldots, P_4$ are 4 generic points of $\mathbb{P}^5$) and show that the ideal of this scheme

$$I_W = q_1^4 \cap \cdots \cap q_5^4 \cap \wp_1^2 \cap \cdots \cap \wp_4^2$$

satisfies $\dim(I_W)_5 = (2^5 - 1) - 23 = 8$.

****

Now, let H be a hyperplane which contains Q_2 through Q_5 but not Q_1; choose P_1' and P_2' generic in H and P_3 and P_4 generic in $\mathbb{P}^5$. Form the (less than generic) scheme:

$$\mathbb{Z} = 4Q_1 + \cdots + 4Q_5 + 2P_1' + 2P_2' + 2P_3 + 2P_4.$$

It would be enough to show that $\dim(I_{\mathbb{Z}})_5 = 8$.

Now add 8 points to $\mathbb{Z}$, call them $\{T_1, \ldots, T_8\}$ - with the first four chosen generically on H and the last four chosen generically in $\mathbb{P}^5$ - and call the resulting scheme $\mathbb{Z}^+$. It will be enough to prove that $\dim_k(I_{\mathbb{Z}^+})_5 = 0$. We want to apply the Lemma to this scheme.

To that end, write

$$Z^+ = \tilde{\mathbb{Z}} + 2P_3 + 2P_4 =$$

$$(4Q_1 + \cdots + 4Q_5 + 2P_1' + 2P_2' + T_1 + \cdots + T_8) + 2P_3 + 2P_4$$

and calculate $Res_H(\tilde{\mathbb{Z}})$ and $Tr_H(\tilde{\mathbb{Z}})$. We find,

$$Res_H(\tilde{\mathbb{Z}}) =$$

$$4Q_1 + 3Q_2 + \cdots + 3Q_5 + P_1' + P_2' + T_5 + \cdots + T_8 \subset \mathbb{P}^5,$$

$$Tr_H(\tilde{\mathbb{Z}}) =$$

$$4Q_2 + \cdots + 4Q_5 + 2P_1' + 2P_2' + T_1 + \cdots + T_4 \subset \mathbb{P}^4 = H.$$

By the Lemma, we choose two generic point P_3', P_4' in H and consider the two schemes:

$$(\mathbb{Z}^+)' = Res_H(\tilde{\mathbb{Z}}) + D_{2,H}(P_3') + D_{2,H}(P_4') \subset \mathbb{P}^5$$

and

$$(\mathbb{Z}^+)'' = Tr_H(\tilde{\mathbb{Z}}) + P_3' + \mathbb{P}_4' \subset \mathbb{P}^4.$$

If we can show that

$$\dim_k(I_{(\mathbb{Z}^+)'})_4 = 0 \text{ and } \dim_k(I_{(\mathbb{Z}^+)''})_5 = 0$$

we will be done.

Now, rewrite $(\mathbb{Z}^+)' = \mathbb{Y} + T_1 + \ldots + T_4 \subset \mathbb{P}^5$, where $\mathbb{Y} =$

$$4Q_1 + 3Q_2 + \cdots + 3Q_5 + D_{2,H}(P_3') + D_{2,H}(P_4') + P_1' + P_2'.$$

Any hypersurface of degree 4 containing that scheme is a cone with vertex Q_1. So, enough to consider the scheme we get by intersecting $\mathbb{Y}$ with a hyperplane of $\mathbb{P}^5$ not through Q_1 and then counting the hypersurfaces of degree 4 that contain the resulting scheme.

We may as well use H. Let $\mathbb{Y}' = \mathbb{Y} \cap H \subset H$, then

$$\mathbb{Y}' = 3Q_2 + \ldots + 3Q_5 + 2P_3' + 2P_4' + P_1' + P_2' \subset H = \mathbb{P}^4.$$

We now apply the induction, and consider $(\mathbb{P}^1)^4$ embedded as $\mathbb{X}_4 \subset \mathbb{P}^{15}$. We know (by induction) that $Sec_1(\mathbb{X}^4)$ has the expected dimension, and so

$$\tilde{W} = 3Q_2 + \ldots + 3Q_5 + 2P_3' + 2P_4'$$

satisfies $\dim_k (I_{\tilde{W}})_4 = 6$.

Consequently, $\dim_k (I_{\mathbb{Y}'})_4 = 6 - 2 = 4$, and since $(\mathbb{Z}^+)'$ is obtained from $\mathbb{Y}$ by adding the four generic points $T_5, \ldots, T_8$, we obtain the desired conclusion that $\dim_k (I_{(Z^+)'})_4 = 0$.

To finish using the lemma we have to also prove that

$$\dim_k (I_{(Z^+)''})_5 = 0.$$

Let's rewrite this subscheme of $\mathbb{P}^4$ as:

$$(Z^+)'' = \widehat{W} + (T_1 + \cdots + T_4)$$

where

$$\widehat{W} = 4Q_2 + \ldots + 4Q_5 + 2P_1' + 2P_2' + P_3' + P_4'.$$

We may as well consider $(I_{\widehat{W}})_5$.

But, for

$$\widehat{W} = 4Q_2 + \ldots + 4Q_5 + 2P_1' + 2P_2' + P_3' + P_4'.$$

we notice that any form of degree 5 vanishing to order 4 at the coordinate points $Q_2, \ldots, Q_5$ of $\mathbb{P}^4$ must have the equation of the hyperplane spanned by those coordinate points as a factor. Call that hyperplane of $\mathbb{P}^4$, H'. We then have

$$\dim_k(I_{\widehat{W}})_5 = \dim_k(I_{Res_{H'}(\widehat{W})})_4.$$

But

$$Res_{H'}(\widehat{W}) = 3Q_2 + \cdots + 3Q_5 + 2P_1' + 2P_2' + P_3' + P_4'$$

(where $P_1', \ldots, P_4'$ were generic in $\mathbb{P}^4$, hence not in H').

But, now we are in exactly the same situation as before and this part of the proof is complete.

For the rest of the argument, we need to show that $Sec_5(\mathbb{X}_5) = \mathbb{P}^{31}$, i.e. we have to show that for the scheme

$$\mathbb{Z} = 4Q_1 + \ldots + 4Q_5 + 2P_1 + \cdots + 2P_6$$

(the P_i generic in $\mathbb{P}^5$) then $(I_{\mathbb{Z}})_5 = 0$.

Again, we choose P_1', P_2', P_3' generically in H (a hyperplane containing $Q_2, \ldots, Q_5$ but not Q_1) and P_4, P_5, P_6 generic in $\mathbb{P}^5$. By the same sort of argument as above (but this time we don't need any T's) we are reduced to showing that a subscheme of $\mathbb{P}^4$ of the form

$$\mathbb{Z}''' = 3Q_2+\ldots+3Q_5+2R_1+2R_2+2R_3+R_4+R_5+R_6$$

(the R_i's generic in $\mathbb{P}^4$) satisfies $(I_{\mathbb{Z}'''})_4 = 0$.

But, the scheme

$$\widetilde{\mathbb{Z}} = 3Q_2 + \ldots + 3Q_5 + 2R_1 + 2R_2 + 2R_3$$

is exactly the scheme we encounter in finding the dimension of $Sec_2(\mathbb{X}_4)$. By induction, we had showed that this dimension was exactly 2. Since we get $\mathbb{Z}'''$ by adding 3 generic points to $\widetilde{\mathbb{Z}}$, the result follows. (Note that in this case there was some "extra" space!)

With this special case of the Theorem you have seen all the basic ingredients of the proof.